Motivic degree zero Donaldson–Thomas invariants

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Abstract Given a smooth complex threefold X, we define the virtual motive $[\mathrm{Hilb}^n(X)]_{\mathrm{vir}}$ of the Hilbert scheme of n points on X. In the case when X is Calabi–Yau, $[\mathrm{Hilb}^n(X)]_{\mathrm{vir}}$ gives a motivic refinement of the n-point degree zero Donaldson–Thomas invariant of X. The key example is $X = \mathbb{C}^3$, where the Hilbert scheme can be expressed as the critical locus of a regular function on a smooth variety, and its virtual motive is defined in terms of the Denef–Loeser motivic nearby fiber. A crucial technical result asserts that if a function is equivariant with respect to a suitable torus action, its motivic nearby fiber is simply given by the motivic class of a general fiber. This allows us to compute the generating function of the virtual motives $[\mathrm{Hilb}^n(\mathbb{C}^3)]_{\mathrm{vir}}$ via a direct computation involving the motivic class of the commuting variety. We then give a formula for the generating function for arbitrary X as a motivic exponential, generalizing known results in lower dimensions. The weight polynomial specialization leads to a product formula in terms of deformed MacMahon functions, analogous to Göttsche's formula for the Poincaré polynomials of the Hilbert schemes of points on surfaces.

Mathematics Subject Classification 14C05 · 14J30 · 14N35

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1 Introduction

Let Z be a scheme of finite type over $\mathbb{C}$. The *virtual Euler characteristic* of Z is defined to be the topological Euler characteristic, weighted by the integer-valued constructible function ν_Z introduced by the first author [1]:

$$\chi_{\text{vir}}(Z) = \sum_{k \in \mathbb{Z}} k \chi \left(\nu_Z^{-1}(k) \right).$$

Unlike the ordinary Euler characteristic, the virtual Euler characteristic is sensitive to singularities and scheme structure. A *virtual motive* of Z is an element $[Z]_{\text{vir}}$ in a suitably augmented Grothendieck group of varieties (the "ring of motivic weights" $\mathcal{M}_{\mathbb{C}}$, see Sect. 2.1) such that

$$\chi \left([Z]_{\text{vir}} \right) = \chi_{\text{vir}}(Z).$$

In the context where Z is a moduli space of sheaves on a Calabi–Yau threefold X, the virtual Euler characteristic $\chi_{\text{vir}}(Z)$ is a (numerical) Donaldson–Thomas invariant. In this setting, we say that $[Z]_{\text{vir}}$ is a *motivic Donaldson–Thomas invariant*.

In this paper, we construct a natural virtual motive $[\text{Hilb}^n(X)]_{\text{vir}}$ for the Hilbert scheme of n points on a smooth threefold X. In the Calabi–Yau case, the virtual Euler characteristics of the Hilbert schemes of points are the degree zero Donaldson–Thomas invariants of X defined in [27] and computed in [2, 23, 24] (cf. Remark 4.8). So we call our virtual motives $[\text{Hilb}^n(X)]_{\text{vir}}$ the motivic degree zero Donaldson–Thomas invariants of X.

If $f : M \to \mathbb{C}$ is a regular function on a smooth variety and

$$Z = \{df = 0\}$$

is its scheme theoretic degeneracy locus, then there is a natural virtual motive (Definition 2.14) given by

$$[Z]_{\text{vir}} = -\mathbb{L}^{-\frac{\dim M}{2}} [\varphi_f],$$

where $[\varphi_f]$ is the motivic vanishing cycle defined by Denef-Loeser [11, 25] and $\mathbb{L}$ is the Lefschetz motive. (This is similar to Kontsevich and Soibelman's approach to motivic Donaldson–Thomas invariants [21].) The function f is often called a global Chern-Simons functional or super-potential. This setting encompasses many useful cases such as when Z is smooth (by letting $(M, f) = (Z, 0)$) and (less trivially) when Z arises as a moduli space of representations of a quiver equipped with a super-potential. The latter includes $\text{Hilb}^n(\mathbb{C}^3)$ which we show is given as the degeneracy locus of an explicit function $f_n : M_n \to \mathbb{C}$ on a smooth space M_n (see Sect. 3.2).

We prove (Theorem B.1, cf. Propositions 2.11, 2.13, and 2.12) that if $f : M \to \mathbb{C}$ is equivariant with respect to a torus action satisfying certain properties, then the motivic vanishing cycle is simply given by the class of the general fiber minus the class of the central fiber:

$$[\varphi_f] = [f^{-1}(1)] - [f^{-1}(0)].$$

This theorem should be applicable in a wide variety of quiver settings and should make the computation of the virtual motives $[Z]_{\text{vir}}$ tractable by quiver techniques.

Indeed, we apply this to compute the motivic degree zero Donaldson–Thomas partition function

$$Z_X(t) = \sum_{n=0}^{\infty} \left[\text{Hilb}^n(X)\right]_{\text{vir}} t^n$$

in the case when X is $\mathbb{C}^3$. Namely, it is given in Theorem 3.7 as

$$Z_{\mathbb{C}^3}(t) = \prod_{m=1}^{\infty} \prod_{k=0}^{m-1} \left(1 - \mathbb{L}^{k+2-\frac{m}{2}} t^m\right)^{-1}.$$

The virtual motive $[\text{Hilb}^n(\mathbb{C}^3)]_{\text{vir}}$ that we construct via the super-potential f_n has good compatibility properties with respect to the Hilbert-Chow morphism $\text{Hilb}^n(\mathbb{C}^3) \to \text{Sym}^n(\mathbb{C}^3)$. Consequently we are able to use these virtual motives to define virtual motives $[\text{Hilb}^n(X)]_{\text{vir}}$ for the Hilbert scheme of any smooth threefold X (see Sect. 4.1). The now standard technology [6, 14, 16, 18] allows us to express in Theorem 4.3 the motivic degree zero Donaldson–Thomas partition function of any threefold X as a motivic exponential

$$Z_X(-t) = \text{Exp}\left([X]\frac{-\mathbb{L}^{-\frac{3}{2}} t}{(1 + \mathbb{L}^{\frac{1}{2}} t)(1 + \mathbb{L}^{-\frac{1}{2}} t)}\right).$$

See Sect. 2.5 for the definition of Exp.

While the above formula only applies when $\dim(X) = 3$, it fits well with corresponding formulas for $\dim(X) < 3$. In these cases, the Hilbert schemes are smooth and thus have canonical virtual motives which are easily expressed in terms of the ordinary classes $[\text{Hilb}^n(X)]$ in the Grothendieck group. The resulting partition functions have been computed for curves [18] and surfaces [16], and all these results can be expressed (Corollary 4.4) in the single formula

$$Z_X(T) = \text{Exp}\left(T[X]_{\text{vir}} \text{Exp}\left(T\left[\mathbb{P}^{d-2}\right]_{\text{vir}}\right)\right)$$

valid when $d = \dim(X)$ is 0, 1, 2, or 3. Here

$$[X]_{\mathrm{vir}} = \mathbb{L}^{-\frac{d}{2}}[X],$$

also[1]

$$T = (-1)^d t,$$

and the class of a negative dimensional projective space is defined by (4.3). In particular, $[\mathbb{P}^{-1}]_{\mathrm{vir}} = 0$ and $[\mathbb{P}^{-2}]_{\mathrm{vir}} = -1$. The above formula has significance for $\dim X > 3$ as well. It holds whenever the virtual motive of the Hilbert scheme is well defined (which for when $\dim X > 3$, is when the Hilbert scheme is smooth). One could thus regard the above formula as *defining* the virtual motive of the Hilbert scheme of points in *all* dimensions and degrees. See Remarks 4.5 and 4.6.

The weight polynomial specialization of the class of a projective manifold gives its Poincaré polynomial. For example, if X is a smooth projective threefold, we get

$$W\big([X], q^{\frac{1}{2}}\big) = \sum_{d=0}^{6} b_d q^{\frac{d}{2}},$$

where b_d is the degree d Betti number of X. Taking the weight polynomial specialization of the virtual motives of the Hilbert schemes gives a virtual version of the Poincaré polynomials of the Hilbert schemes. In Theorem 4.7 we apply weight polynomials to our formula for $Z_X(t)$ to get

$$\sum_{n=0}^{\infty} W\big([\mathrm{Hilb}^n(X)]_{\mathrm{vir}}, q^{\frac{1}{2}}\big) t^n = \prod_{d=0}^{6} M_{\frac{d-3}{2}}\big(-t, -q^{\frac{1}{2}}\big)^{(-1)^d b_d}$$

where M_δ is the q-deformed MacMahon function

$$M_\delta\big(t, q^{\frac{1}{2}}\big) = \prod_{m=1}^{\infty} \prod_{k=0}^{m-1} \big(1 - q^{k+\frac{1}{2}-\frac{m}{2}+\delta} t^m\big)^{-1}.$$

The above formula is the analog for threefolds of Göttsche's famous product formula [15] for the Poincaré polynomials of Hilbert schemes on surfaces. Similar q-deformed MacMahon functions appear in the refined topological vertex of Iqbal-Kozcaz-Vafa [19]. We discuss M_δ further in Appendix A.

In addition to motivic Donaldson–Thomas invariants, one may consider *categorified Donaldson–Thomas invariants*. Such a categorification is a lift

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[1] Note that the operator Exp depends on the variable t. In particular, one cannot simply substitute t for T in the above equation for Z_X. See Remark 2.8.

of the numerical Donaldson–Thomas invariant $\chi_{\mathrm{vir}}(Z)$ to an object $[Z]_{cat}$ in a category with a cohomological functor $H^\bullet$ such that

$$\chi\big(H^\bullet([Z]_{cat})\big) = \chi_{\mathrm{vir}}(Z).$$

We have partial results toward constructing categorified degree zero Donaldson–Thomas invariants which we discuss in Sect. 4.4. See also the recent work [22].

2 Motivic weights and vanishing cycles

All our varieties and maps are defined over the field $\mathbb{C}$ of complex numbers.

2.1 The ring of motivic weights

Let $K_0(\mathrm{Var}_{\mathbb{C}})$ be the $\mathbb{Z}$-module generated by isomorphism classes of reduced $\mathbb{C}$-varieties,[2] under the scissor relation

$$[X] = [Y] + [X \setminus Y] \in K_0(\mathrm{Var}_{\mathbb{C}})$$

for $Y \subset X$ a closed subvariety. $K_0(\mathrm{Var}_{\mathbb{C}})$ has a ring structure whose product is the Cartesian product of varieties. The following two properties of the setup are well known.

1. If $f : X \to S$ is a Zariski locally trivial fibration with fiber F, then

$$[X] = [S] \cdot [F] \in K_0(\mathrm{Var}_{\mathbb{C}}).$$

2. If $f : X \to Y$ is a bijective morphism, then

$$[X] = [Y] \in K_0(\mathrm{Var}_{\mathbb{C}}).$$

Let

$$\mathbb{L} = \big[\mathbb{A}^1\big] \in K_0(\mathrm{Var}_{\mathbb{C}})$$

be the class of the affine line. We define the *ring of motivic weights* (or *motivic ring* for short) to be

$$\mathcal{M}_{\mathbb{C}} = K_0(\mathrm{Var}_{\mathbb{C}})\big[\mathbb{L}^{-\frac{1}{2}}\big].$$

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[2]One can also consider the rings $K_0(Sch_{\mathbb{C}})$ and $K_0(Sp_{\mathbb{C}})$ generated by schemes or algebraic spaces (of finite type) with the same relations. By [5, Lemma 2.12], they are all the same. In particular, for X a scheme, its class in $K_0(\mathrm{Var}_{\mathbb{C}})$ is given by $[X_{\mathrm{red}}]$, the class of the associated reduced scheme. We will implicitly use this identification throughout the paper without further comment.

We set up notation for some elements of $\mathcal{M}_{\mathbb{C}}$ that we will use later. Let

$$[n]_{\mathbb{L}}! = (\mathbb{L}^n - 1)(\mathbb{L}^{n-1} - 1) \cdots (\mathbb{L} - 1)$$

and let

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathbb{L}} = \frac{[n]_{\mathbb{L}}!}{[n-k]_{\mathbb{L}}![k]_{\mathbb{L}}!}.$$

Using property (1) above, elementary arguments show that

$$[\mathrm{GL}_n] = \mathbb{L}^{\binom{n}{2}}[n]_{\mathbb{L}}!$$

Then using the elementary identity $\binom{n}{2} - \binom{k}{2} - \binom{n-k}{2} = (n-k)k$, the class of the Grassmannian is easily derived:

$$[Gr(k,n)] = \begin{bmatrix} n \\ k \end{bmatrix}_{\mathbb{L}}.$$

For later reference, we recall the computation of the motivic weight of the stack of pairs of commuting matrices. Let V_n be an n-dimensional vector space, and let

$$C_n \subset \mathrm{Hom}(V_n, V_n)^{\times 2}$$

denote the (reduced) variety of pairs of commuting $n \times n$ matrices over the complex numbers. Let

$$\widetilde{c}_n = \frac{[C_n]}{[\mathrm{GL}_n]} \in \mathcal{M}_{\mathbb{C}}\big[(1 - \mathbb{L}^n)^{-1} : n \geq 1\big] \tag{2.1}$$

be its class, renormalized by taking account of the global symmetry group GL_n. Consider the generating series

$$C(t) = \sum_{n \geq 0} \widetilde{c}_n t^n. \tag{2.2}$$

Proposition 2.1 *We have*

$$C(t) = \prod_{m=1}^{\infty} \prod_{j=0}^{\infty} \big(1 - \mathbb{L}^{1-j} t^m\big)^{-1}. \tag{2.3}$$

Proof The main result of the paper of Feit and Fine [13] is the analogous formula

$$C(t,q) = \prod_{m=1}^{\infty} \prod_{j=0}^{\infty} \big(1 - q^{1-j} t^m\big)^{-1}$$

for the generating series of the number of pairs of commuting matrices over the finite field $\mathbb{F}_q$, renormalized as above. Feit and Fine's method is motivic; in essence they provide an affine paving of C_n. For details, see [28]. $\qquad\square$

Remark 2.2 In (2.2), the coefficient $\widetilde{c}_n$ of t^n in $C(t)$ is in the ring $\mathcal{M}_\mathbb{C}[(1 - \mathbb{L}^n)^{-1}\colon n \geq 1]$. In (2.3), the coefficients are the Laurent expansions in $\mathbb{L}$ of these elements.

The following result is now standard [16, Lemma 4.4].

Lemma 2.3 *Let Z be a variety with the free action of a finite group G. Extend the action of G to $Z \times \mathbb{A}^n$ using a linear action of G on the second factor. Then the motivic weights of the quotients are related by*

$$\big[(Z \times \mathbb{A}^n)/G\big] = \mathbb{L}^n[Z/G] \in \mathcal{M}_\mathbb{C}.$$

Proof Let $\pi\colon (Z \times \mathbb{A}^n)/G \to Z/G$ be the projection. By assumption, π is étale locally trivial with fiber $\mathbb{A}^n$, and with linear transition maps. Thus it is an étale vector bundle on Z/G. But then by Hilbert's Theorem 90, it is Zariski locally trivial. $\qquad\square$

2.2 Homomorphisms from the ring of motivic weights

The ring of motivic weights admits a number of well-known ring homomorphisms. Deligne's mixed Hodge structure on compactly supported cohomology of a variety X gives rise to the E-polynomial homomorphism

$$E\colon K_0(\mathrm{Var}_\mathbb{C}) \to \mathbb{Z}[x, y]$$

defined on generators by

$$E\big([X]; x, y\big) = \sum_{p,q} x^p y^q \sum_i (-1)^i \dim H^{p,q}\big(H_c^i(X, \mathbb{Q})\big).$$

This extends to a ring homomorphism

$$E\colon \mathcal{M}_\mathbb{C} \to \mathbb{Z}\big[x, y, (xy)^{-\frac{1}{2}}\big]$$

by defining

$$E\big(\mathbb{L}^n\big) = (xy)^n$$

for half-integers n.

The weight polynomial homomorphism

$$W\colon \mathcal{M}_\mathbb{C} \to \mathbb{Z}\big[q^{\pm\frac{1}{2}}\big],$$

is defined by the specialization

$$x = y = -q^{\frac{1}{2}}, \qquad (xy)^{\frac{1}{2}} = q^{\frac{1}{2}}.$$

This maps $\mathbb{L}$ to q, and encodes the dimensions of the graded quotients of the compactly supported cohomology of X under the weight filtration, disregarding the Hodge filtration. For smooth projective X, $W([X]; q^{\frac{1}{2}})$ is simply the Poincaré polynomial of X. Specializing further,

$$\chi([X]) = W([X]; q^{\frac{1}{2}} = -1)$$

defines the map

$$\chi : \mathcal{M}_{\mathbb{C}} \to \mathbb{Z}$$

of compactly supported Euler characteristic; this agrees with the ordinary Euler characteristic.

2.3 Relative motivic weights

Given a reduced (but not necessarily irreducible) variety S, let $K_0(\mathrm{Var}_S)$ be the $\mathbb{Z}$-module generated by isomorphism classes of (reduced) S-varieties, under the scissor relation for S-varieties, and ring structure whose multiplication is given by fiber product over S. Elements of this ring will be denoted $[X]_S$. A morphism $f : S \to T$ induces a ring homomorphism $f^* : K_0(\mathrm{Var}_T) \to K_0(\mathrm{Var}_S)$ given by fiber product. In particular, $K_0(\mathrm{Var}_S)$ is always a $K_0(\mathrm{Var}_{\mathbb{C}})$-algebra. Thus we can let

$$\mathcal{M}_S = K_0(\mathrm{Var}_S)\left[\mathbb{L}^{-\frac{1}{2}}\right],$$

an $\mathcal{M}_{\mathbb{C}}$-module. A morphism $f : S \to T$ induces a ring homomorphism $f^* : \mathcal{M}_T \to \mathcal{M}_S$ by pullback, and a direct image homomorphism $f_! : \mathcal{M}_S \to \mathcal{M}_T$ by composition, the latter a map of $\mathcal{M}_T$-modules.

In the relative case, the E-polynomial, weight polynomial and Euler characteristic specializations map to the K-group of variations of mixed Hodge structures (or mixed Hodge modules), the K-group of mixed sheaves, and the space of constructible functions, respectively.

2.4 Equivariant motivic weights

Let G be a finite group. An action of G on a variety X is said to be good, if every point of X is contained in an affine G-invariant open subset. Actions of finite groups on quasiprojective varieties are thus good.

We will have occasion to use two versions of equivariant rings of motivic weights. For a fixed variety S with good G-action, let $\widetilde{K}_0^G(\mathrm{Var}_S)$ be the K-group generated by good G-varieties over S, modulo the G-scissor relation. Let also $K_0^G(\mathrm{Var}_S)$ be the quotient of $\widetilde{K}_0^G(\mathrm{Var}_S)$ by the further relations

$$[V] = \left[\mathbb{C}^r \times S\right] \tag{2.4}$$

where $V \to S$ is any G-equivariant vector bundle over S of rank r and $\mathbb{C}^r \times S$ is the trivial rank r bundle with trivial G-action. The affine S-line $\mathbb{A}^1 \times S$ inherits a G-action and so defines elements $\mathbb{L} \in \widetilde{K}_0^G(\mathrm{Var}_S)$ and $\mathbb{L} \in K_0^G(\mathrm{Var}_S)$; we let $\widetilde{\mathcal{M}}_S^G$ and $\mathcal{M}_S^G$ be the corresponding extensions by $\mathbb{L}^{-\frac{1}{2}}$.

If the G-action on S is trivial, then $\mathcal{M}_S^G$ and $\widetilde{\mathcal{M}}_S^G$ are $\mathcal{M}_S$-algebras, using the map which regards a motive over S as a G-motive with trivial action. In this case, there is a map of $\mathcal{M}_S$-modules

$$\pi_G : \widetilde{\mathcal{M}}_S^G \to \mathcal{M}_S \tag{2.5}$$

given on generators by taking the orbit space. This operation is clearly compatible with the module operations and scissor relation. In general, this map does not respect the relations (2.4), so it does not descend to $\mathcal{M}_S^G$.

As a variant of this construction, let $\hat{\mu} = \lim_{\leftarrow} \mu_n$ be the group of roots of unity. A good $\hat{\mu}$-action on a variety X is one where $\hat{\mu}$ acts via a finite quotient and that action is good. Let $\mathcal{M}_S^{\hat{\mu}} = K_0^{\hat{\mu}}(\mathrm{Var}_S)[\mathbb{L}^{-\frac{1}{2}}]$ be the corresponding K-group, incorporating the relations (2.4). The additive group $\mathcal{M}_S^{\hat{\mu}}$ can be endowed with an associative operation $\star$ using convolution involving the classes of Fermat curves [9, 21, 25]. This product agrees with the ordinary (direct) product on the subalgebra $\mathcal{M}_S \subset \mathcal{M}_S^{\hat{\mu}}$ of classes with trivial $\hat{\mu}$-action, but not in general.

We will need the following statement below.

Lemma 2.4 *For any positive integer n, there exists a well-defined nth power map*

$$(-)^n : \mathcal{M}_{\mathbb{C}} \to \widetilde{\mathcal{M}}_{\mathbb{C}}^{S_n}$$

to the ring of S_n-equivariant motivic weights, defined by the property that for a class $A \in \mathcal{M}_{\mathbb{C}}$ represented by a quasiprojective varity, A^n as an equivariant motive is the class of the ordinary nth power of that variety, carrying the standard S_n-action.

Proof Assume first that we can write $A = B \pm C$ where B, C are classes represented by quasiprojective varieties. Then by the binomial theorem,

$$A^n = \sum_{i=0}^{n} (\pm 1)^i X_i,$$

where X_i is the variety which consists of $\binom{n}{i}$ disjoint copies of the variety $B^{n-i} C^i$. We claim that all these varieties X_i carry geometric S_n-actions. Label the B's and C's in the expansion of $(B \pm C)^n$ with the labels $1, \ldots, n$, depending on which bracket they come from. Then every term will carry exactly one instance of each label. The group S_n acts by interchanging the labels, and does not change the number of B's and C's in a monomial. Thus each element of S_n defines a map $X_i \to X_i$, i.e. a geometric automorphism of X_i, which is moreover good. The class A^n thus becomes an element of $\widetilde{\mathcal{M}}_{\mathbb{C}}^{S_n}$.

The extension of this argument to the case of more than two summands is clear. Since up to powers of $\mathbb{L}^{\pm \frac{1}{2}}$, every class in $\mathcal{M}_{\mathbb{C}}$ is a signed sum of classes represented by quasiprojective varieties, this shows that every A^n can be viewed as an element of $\widetilde{\mathcal{M}}_{\mathbb{C}}^{S_n}$.

Next, suppose that $A = B + C$ with A, B, C all represented by quasiprojective varieties. Then A^n is already an element of $\widetilde{\mathcal{M}}_{\mathbb{C}}^{S_n}$, and the above definition gives another way of viewing $(B + C)^n$ as an equivariant class. However, in this case the above X_i simply give an S_n-equivariant stratification of the variety underlying A^n, and thus the two equivariant classes are the same. Again, the extension to more than two summands is clear.

Finally, if $A = B - C = D - E$ with B, C, D, E quasiprojective, then $B + E = C + D$. Thus there exists a common refinement, a finite set of classes $\{A_i : i \in I\}$ represented by quasiprojective varieties, and subsets $J, K \subset I$ such that $B = \sum_{i \in J} A_i$, $C = \sum_{i \in K} A_i$, $D = \sum_{i \in I \setminus K} A_i$, $E = \sum_{i \in I \setminus J} A_i$. So the two differences $B - C$ and $D - E$ are represented by the same signed sum of A_is and so the two definitions of $A^n \in \widetilde{\mathcal{M}}_{\mathbb{C}}^{S_n}$ are equal. □

2.5 Power structure on the ring of motivic weights

Recall that a power structure on a ring R is a map

$$\left(1 + t R[[t]]\right) \times R \to 1 + t R[[t]]$$
$$\left(A(t), m\right) \mapsto A(t)^m$$

satisfying $A(t)^0 = 1$, $A(t)^1 = A(t)$, $A(t)^{m+n} = A(t)^m A(t)^n$, $A(t)^{mn} = (A(t)^m)^n$, $A(t)^m B(t)^m = (A(t) B(t))^m$, as well as $(1+t)^m = 1 + mt + O(t^2)$.

Theorem 2.5 (Gusein-Zade et al. [17], cf. Getzler [14]) *There exists a power structure on the Grothendieck ring $K_0(\mathrm{Var}_\mathbb{C})$, defined uniquely by the property that for a variety X,*

$$(1-t)^{-[X]} = \sum_{n=0}^{\infty} [\mathrm{Sym}^n X] t^n$$

is the generating function of symmetric products of X, its motivic zeta function.

Since we will need it below, we recall the definition. Let $A(t) = 1 + \sum_{i \geq 1} A_i t^i$ be a series with $A_i \in K_0(\mathrm{Var}_\mathbb{C})$. Then for $[X] \in K_0(\mathrm{Var}_\mathbb{C})$ a class represented by a variety X, the definition of [17] reads

$$A(t)^{[X]} = 1 + \sum_\alpha \pi_{G_\alpha} \left[\left(\prod_i X^{\alpha_i} \setminus \Delta \right) \cdot \left(\prod_i A_i^{\alpha_i} \right) \right] t^{|\alpha|}. \qquad (2.6)$$

Here the summation runs over all partitions α; for a partition α, write α_i for the number of parts of size i, and let $G_\alpha = \prod_i S_{\alpha_i}$ denote the standard product of symmetric groups. Δ denotes the big diagonal in any product of copies of the variety X. By Lemma 2.4, the product $(\prod_i X^{\alpha_i} \setminus \Delta) \times \prod_i A_i^{\alpha_i}$ can be represented by a class in $K_0^{G_\alpha}(\mathrm{Var}_\mathbb{C})$, and the map π_{G_α} is the quotient map (2.5).

Note that if we replace the coefficients A_i by $\mathbb{L}^{c_i} A_i$ for positive integers c_i, then by Lemma 2.3 above, the individual terms in the sum change as

$$\pi_{G_\alpha} \left[\left(\prod_i X^{\alpha_i} \setminus \Delta \right) \times \prod_i (\mathbb{L}^{c_i} A_i)^{\alpha_i} \right]$$
$$= \mathbb{L}^{\sum_i c_i \alpha_i} \pi_{G_\alpha} \left[\left(\prod_i X^{\alpha_i} \setminus \Delta \right) \times \prod_i A_i^{\alpha_i} \right],$$

since the action of G_α on the $\mathbb{L}^{\sum_i c_i \alpha_i}$ factor comes from a product of permutation actions and is hence linear. We thus get the substitution rule

$$A(\mathbb{L}^c t)^{[X]} = A(t)^{[X]}|_{t \mapsto \mathbb{L}^c t}$$

for positive integer c. We extend the definition (2.6) to allow coefficients A_i which are from $\mathcal{M}_\mathbb{C}$, by the formula

$$\pi_{G_\alpha} \left[\left(\prod_i X^{\alpha_i} \setminus \Delta \right) \times \prod_i ((-\mathbb{L}^{\frac{1}{2}})^{c_i} A_i)^{\alpha_i} \right]$$

$$= \left(-\mathbb{L}^{\frac{1}{2}}\right)^{\sum_i c_i \alpha_i} \pi_{G_\alpha} \left[\left(\prod_i X^{\alpha_i} \setminus \Delta \right) \times \prod_i A_i^{\alpha_i} \right]$$

for integers c_i (see Remark 2.7 for the reason for the appearance of signs here). This implies the substitution rule

$$A\left(\left(-\mathbb{L}^{\frac{1}{2}}\right)^n t\right)^{[X]} = A(t)^{[X]}\Big|_{t \mapsto (-\mathbb{L}^{\frac{1}{2}})^n t}$$

for integers n. We also extend the power structure to exponents from the ring $\mathcal{M}_\mathbb{C}$ by defining

$$(1-t)^{-\left(-\mathbb{L}^{\frac{1}{2}}\right)^n [X]} = \left(1 - \left(-\mathbb{L}^{\frac{1}{2}}\right)^n t\right)^{-[X]}$$

for all $n \in \mathbb{Z}$; as in Theorem 2.5, this determines a unique extension of the power structure.

Finally, still following [14, 17], introduce the map

$$\mathrm{Exp}\colon t\mathcal{M}_\mathbb{C}\big[[t]\big] \to 1 + t\mathcal{M}_\mathbb{C}\big[[t]\big]$$

by

$$\mathrm{Exp} \sum_{n=1}^{\infty} [A_n] t^n = \prod_{n \geq 1} \left(1 - t^n\right)^{-[A_n]}. \tag{2.7}$$

Exp is an isomorphism from the additive group $t\mathcal{M}_\mathbb{C}[[t]]$ to the multiplicative group $1 + t\mathcal{M}_\mathbb{C}[[t]]$.

The above equations imply the following substitution rule:

$$\mathrm{Exp}\big(A(t)\big)\Big|_{t \mapsto (-\mathbb{L}^{\frac{1}{2}})^n t} = \mathrm{Exp}\big(A\big(\left(-\mathbb{L}^{\frac{1}{2}}\right)^n t\big)\big). \tag{2.8}$$

Example 2.6 It is easy to check that the generating series $C(t)$ of the motivic weight of pairs of commuting matrices (2.3) can be written as a motivic exponential (cf. [22, Proposition 7]):

$$C(t) = \mathrm{Exp}\left(\frac{\mathbb{L}^2}{\mathbb{L}-1} \frac{t}{1-t}\right).$$

Remark 2.7 The existence a power structure on $K_0(\mathrm{Var}_\mathbb{C})$ is closely related to the fact that $K_0(\mathrm{Var}_\mathbb{C})$ has the structure of a pre-λ-ring where the operations σ_n are characterized by $\sigma_n(X) = [\mathrm{Sym}^n(X)]$ (see [14]). In order to extend the power structure to $\mathcal{M}_\mathbb{C}$ so that the Euler characteristic homomorphism

respects the power structure, we must have[3] $\sigma_n(-\mathbb{L}^{\frac{1}{2}}) = (-\mathbb{L}^{\frac{1}{2}})^n$ which explains the signs in the formulae above.

Remark 2.8 Note that the isomorphism Exp does not commute with substitutions in the variable t. For example, $\text{Exp}(t) = \frac{1}{1-t}$ whereas $\text{Exp}(-t) = 1 - t$. This explains why we use the auxilary variable T in the introduction.

2.6 Motivic nearby and vanishing cycles

Let
$$f : X \to \mathbb{C}$$
be a regular function on a smooth variety X, and let $X_0 = f^{-1}(0)$ be the central fiber.

Using arc spaces, Denef and Loeser [11, §3],[25, §5] define $[\psi_f]_{X_0} \in \mathcal{M}_{X_0}^{\hat{\mu}}$, the relative motivic nearby cycle of f. Using motivic integration, Denef–Loeser give an explicit formula for $[\psi_f]$ in terms of an embedded resolution of f. We give this formula in detail in Appendix B.

Let
$$[\varphi_f]_{X_0} = [\psi_f]_{X_0} - [X_0]_{X_0} \in \mathcal{M}_{X_0}^{\hat{\mu}}$$
be the relative motivic vanishing cycle of f. It follows directly from the definitions that over the smooth locus of the central fiber, the classes $[\psi_f]_{X_0}$ and $[X_0]_{X_0}$ coincide, so the motivic difference $[\varphi_f]_{X_0}$ is a relative class $[\varphi_f]_{\text{Sing}(X_0)}$ over the singular locus of X_0. The latter is exactly the degeneracy locus $Z \subset X$, the subscheme of X_0 given by the equations $\{df = 0\}$. We will denote by $[\psi_f], [\varphi_f] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ the absolute motivic nearby and vanishing cycles, the images of the relative classes under pushforward to the point.

We next recall the motivic Thom–Sebastiani theorem. Given two regular functions $f : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$ on smooth varieties X, Y, define the function $f \oplus g : X \times Y \to \mathbb{C}$ by
$$(f \oplus g)(x, y) = f(x) + g(y).$$

Theorem 2.9 (Denef–Loeser [10], Looijenga [25]) *Let f, g be non-constant regular functions on smooth varieties X, Y, and let X_0, Y_0 be their zero fibers. Let*
$$i : X_0 \times Y_0 \to (X \times Y)_0$$

---

[3]We thank Sven Meinhardt for calling our attention to this sign issue.

denote the natural inclusion into the zero fiber of $f \oplus g$. Then

$$i^*[-\varphi_{f\oplus g}]_{X_0\times Y_0} = p_X^*[-\varphi_f]_{X_0} \star p_Y^*[-\varphi_g]_{Y_0} \in \mathcal{M}_{X_0\times Y_0}^{\hat{\mu}}$$

where p_X, p_Y are the projections from $X_0 \times Y_0$ to the two factors.

Remark 2.10 Consider the functions $f(x) = x^2$ and $g(y) = y^2$. Restricting to the origins in $\mathbb{C}$ and $\mathbb{C}^2$, Theorem 2.9 reads

$$(-\varphi_{x^2}) \star (-\varphi_{y^2}) = -\varphi_{x^2+y^2} \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}.$$

Direct computation (using for example (B.1)) yields

$$-\varphi_{x^2+y^2} = \mathbb{L}, \quad -\varphi_{x^2} = -\varphi_{y^2} = 1 - [2\text{pt}, \mu_2]$$

where $[2\text{pt}, \mu_2]$ is the space of 2 points, with the μ_2-action which swaps the points. We see that rather than adjoining $\mathbb{L}^{\frac{1}{2}}$ formally to $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$, we could have taken

$$\mathbb{L}^{\frac{1}{2}} = 1 - [2\text{pt}, \mu_2] \tag{2.9}$$

(cf. [21, Remark 19]). Indeed, imposing the above equation as a relation in $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ has some desirable consequences such as making the relative virtual motive of a smooth variety canonical at each point; see Remark 2.15.[4]

2.7 Torus-equivariant families

We wish to study regular functions $f : X \to \mathbb{C}$ on smooth varieties X with the following equivariance property.[5] We assume there exists an action of a connected complex torus T on X such that f is T-equivariant with respect to a primitive character $\chi : T \to \mathbb{C}^*$. That is, for all $t \in T$ and $x \in X$, we have $f(tx) = \chi(t)f(x)$.

Assuming the existence of such a T-action, the family defined by f is trivial away from the central fiber. Indeed, since χ is primitive, there exists a 1-parameter subgroup $\mathbb{C}^* \subset T$ such that χ is an isomorphism restricted to $\mathbb{C}^*$. Let $X_1 = f^{-1}(1)$, then the map $X_1 \times \mathbb{C}^* \to X - X_0$ given by $(x, \lambda) \mapsto \lambda \cdot x$ has inverse

$$x \mapsto \left(\frac{1}{f(x)} \cdot x, f(x) \right)$$

and thus defines an isomorphism $X_1 \times \mathbb{C}^* \cong X - X_0$.

---

[4]We thank Sheldon Katz for discussions on this issue.

[5]We thank Patrick Brosnan and Jörg Schürmann for very helpful correspondence on this subject.

Proposition 2.11 *Assume that the regular function $f : X \to \mathbb{C}$ on a smooth variety X has a T-action as above, and assume that f is proper. Then the absolute motivic vanishing cycle $[\varphi_f]$ of f lies in the subring $\mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$. Moreover, this class can be computed as the motivic difference*

$$[\varphi_f] = [X_1] - [X_0] \in \mathcal{M}_{\mathbb{C}}$$

of the general and central fibers of f.

Proof Using the trivialization of the family discussed above, there is a diagram

$$
\begin{array}{ccc}
X & \dashrightarrow & X_1 \times \mathbb{C} \\
& f \searrow \quad \swarrow p & \\
& \mathbb{C} &
\end{array}
$$
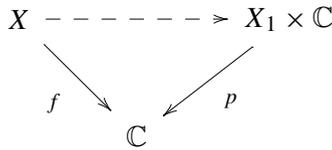
with the birational map being an isomorphism over $\mathbb{C}^*$; here X_t denotes the fiber of f over $t \in \mathbb{C}$, and p denotes the projection to the second factor.

The fiber product W of f and p is proper over $\mathbb{C}$. Let $\bar{Z}$ be the irreducible component of the closure of the graph Γ_g which maps dominantly to $\mathbb{C}$; $\bar{Z}$ is proper over the fiber product W so proper and birational over X and $X_1 \times \mathbb{C}$. Let Z be a desingularization of $\bar{Z}$. We get a diagram

$$
\begin{array}{ccc}
& Z & \\
g \swarrow & & \searrow h \\
X \dashrightarrow & & X_1 \times \mathbb{C} \\
f \searrow & & \swarrow p \\
& \mathbb{C} &
\end{array}
$$
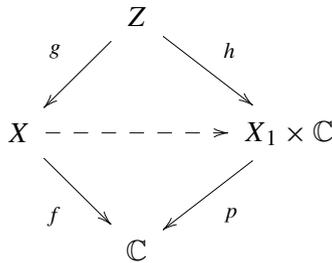
with g, h proper maps.

Denote the composite $f \circ g = p_2 \circ h$ by k. On the central fibers, we get a diagram

$$
\begin{array}{ccc}
& Z_0 & \\
g_0 \swarrow & & \searrow h_0 \\
X_0 & & X_1 \\
f_0 \searrow & & \swarrow p_0 \\
& \{0\} &
\end{array}
$$
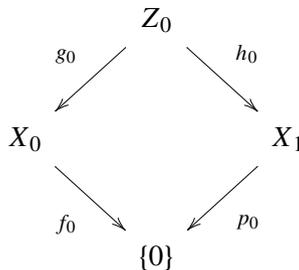
since the central fiber of the family p is X_1; of course the central fibers are no longer birational necessarily.

By [4, Remark 2.7], for the motivic relative nearby cycles,

$$[\psi_f]_{X_0} = g_{0!}[\psi_k]_{Z_0} \in \mathcal{M}_{X_0}^{\hat{\mu}},$$

and

$$[\psi_p]_{X_1} = h_{0!}[\psi_k]_{Z_0} \in \mathcal{M}_{X_1}^{\hat{\mu}}.$$

Thus, the absolute motivic nearby cycle of f is given by

$$[\psi_f] = f_{0!}g_{0!}[\psi_k]_{Z_0} = k_{0!}[\psi_k]_{Z_0} = p_{0!}[\psi_p]_{X_1} \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}.$$

But p is an algebraically trivial proper family over $\mathbb{C}$, so its motivic nearby cycle is the class of its central fiber with trivial monodromy. So the absolute motivic nearby cycle of f is

$$[\psi_f] = p_{0!}[X_1]_{X_1} = [X_1] \in \mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}.$$

Finally by definition,

$$[\varphi_f] = [\psi_f] - [X_0],$$

with X_0 carrying the trivial $\hat{\mu}$-action. The proof is complete. $\qquad\square$

In our examples, our f will not be proper. To weaken this assumption, we say that an action of $\mathbb{C}^*$ on a variety V is *circle compact*, if the fixed point set $V^{\mathbb{C}^*}$ is compact and moreover, for all $v \in V$, the limit $\lim_{\lambda \to 0} \lambda \cdot y$ exists. We use the following variant of Proposition 2.11.

Proposition 2.12 *Let $f : X \to \mathbb{C}$ be a regular function on a smooth quasi-projective complex variety. Suppose that T is a connected complex torus with a linearized action on X such that f is T-equivariant with respect to a primitive character χ, i.e. $f(tx) = \chi(t)f(x)$ for all $t \in T$ $x \in X$. Moreover, suppose that there exists $\mathbb{C}^* \subset T$ such that the induced $\mathbb{C}^*$-action on X is circle compact. Then the absolute motivic vanishing cycle $[\varphi_f]$ of f lies in the subring $\mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$, and it can be expressed as the motivic difference*

$$[\varphi_f] = [X_1] - [X_0] \in \mathcal{M}_{\mathbb{C}}$$

of the general and central fibers of f.

We do not have a conceptual proof of this Proposition as we did for Proposition 2.11. Instead, in Appendix B, we prove this directly using Denef and Loeser's motivic integration formula for $[\varphi_f]$. The key point is that the circle

compact $\mathbb{C}^*$-action gives rise to a Białynicki-Birula stratification of X. The conditions that X is quasi-projective and the T-action is linear can probably weakened; they are added for convenience in the proof and because they should hold in most cases of interest. The condition that the fixed point set of the $\mathbb{C}^*$-action is compact can be dropped; we only use the existence of $\lambda \to 0$ limits. For a recent significant generalization of this result, see [8, Theorem 1.3].

We will also use the following enhancement of the proposition.

Proposition 2.13 *Let $f : X \to \mathbb{C}$ be a T-equivariant regular function satisfying the assumptions of Proposition* 2.12. *Let $Z = \{df = 0\}$ be the degeneracy locus of f and let $Z_{\mathrm{aff}} \subset X_{\mathrm{aff}}$ be the affinizations of Z and X. Suppose that $X_0 = f^{-1}(0)$ is reduced. Then $[\varphi_f]_{Z_{\mathrm{aff}}}$, the motivic vanishing cycle of f relative to Z_{aff}, lies in the subring $\mathcal{M}_{Z_{\mathrm{aff}}} \subset \mathcal{M}_{Z_{\mathrm{aff}}}^{\hat{\mu}}$.*

This result will also be proved in Appendix B.

2.8 The virtual motive of a degeneracy locus

Definition 2.14 Let $f : X \to \mathbb{C}$ be a regular function on a smooth variety X, and let

$$Z = \{df = 0\} \subset X$$

be its degeneracy locus. We define the *relative virtual motive* of Z to be

$$[Z]_{\mathrm{relvir}} = -\mathbb{L}^{-\frac{\dim X}{2}} [\varphi_f]_Z \in \mathcal{M}_Z^{\hat{\mu}},$$

and the *absolute virtual motive* of Z to be

$$[Z]_{\mathrm{vir}} = -\mathbb{L}^{-\frac{\dim X}{2}} [\varphi_f] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}},$$

the pushforward of the relative virtual motive $[Z]_{\mathrm{relvir}}$ to the absolute motivic ring $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$.

Remark 2.15 As a degenerate but important example, consider $f = 0$. Then we have $X_0 = X$ and $[\psi_f]_{X_0} = 0$, so the virtual motives of a smooth variety X with $f = 0$ are given by

$$[X]_{\mathrm{relvir}} = \mathbb{L}^{-\frac{\dim X}{2}} [X]_X \in \mathcal{M}_X \subset \mathcal{M}_X^{\hat{\mu}} \tag{2.10}$$

and

$$[X]_{\mathrm{vir}} = \mathbb{L}^{-\frac{\dim X}{2}} [X] \in \mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}. \tag{2.11}$$

If one imposes equation (2.9) as a relation in $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$, then it is not hard to show that whenever Z is smooth, $[Z]_{\mathrm{relvir}}$ agrees with equation (2.10) at each point, that is for each $P \in Z$, $[Z]_{\mathrm{relvir}}|_P = \mathbb{L}^{-\frac{\dim Z}{2}}$.

Proposition 2.16

1. *At a point $P \in Z$, the fiberwise Euler characteristic of the relative virtual motive $[Z]_{\mathrm{relvir}} \in \mathcal{M}_Z^{\hat{\mu}}$, evaluated at the specialization $\mathbb{L}^{\frac{1}{2}} = -1$, is equal to the value at $P \in Z$ of the constructible function ν_Z of [1].*
2. *The Euler characteristic of the absolute virtual motive $[Z]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ is the virtual Euler characteristic $\chi_{\mathrm{vir}}(Z) \in \mathbb{Z}$ of [1]:*

$$\chi\big([Z]_{\mathrm{vir}}\big) = \chi_{\mathrm{vir}}(X) = \sum_{k \in \mathbb{Z}} k \chi\big(\nu_Z^{-1}(k)\big).$$

Proof By [1, Eq. (4)], for $Z = \{df = 0\} \subset X$, the value of the function ν_Z at the point P is

$$\nu_Z(P) = (-1)^{\dim X}\big(1 - \chi(F_P)\big),$$

where F_P is the Milnor fiber of f at P. On the other hand, the pointwise Euler characteristic of $[\varphi_f]$ at P is the Euler characteristic of the reduced cohomology of the Milnor fiber F_P [11, Theorem 3.5.5], equal to $\chi(F_P) - 1$. The factor $-\mathbb{L}^{\dim X/2}$ at $\mathbb{L}^{\frac{1}{2}} = -1$ contributes the factor $-(-1)^{\dim X}$. This proves (1). (2) clearly follows from (1). $\qquad\qquad\square$

Remark 2.17 If $Z = \{df = 0\}$ is a moduli space of sheaves on a Calabi–Yau threefold, then the associated Donaldson–Thomas invariant is given by $\chi_{\mathrm{vir}}(Z)$. So by the above proposition, $[Z]_{\mathrm{vir}}$ is a motivic refinement of the Donaldson–Thomas invariant and hence can be regarded as a *motivic Donaldson–Thomas invariant*. The function f in this context is called a *global Chern-Simons functional* or a *super-potential*.

Remark 2.18 Unlike the ordinary motivic class of Z, the virtual motive is sensitive to both the singularities and the scheme structure of Z since in particular, the constructible function ν_Z is. However, unlike the function ν_Z, we expect the virtual motive of Z to depend on its presentation as a degeneracy locus $Z = \{df = 0\}$ and not just its scheme structure. We will not include the pair (X, f) in the notation but it will be assumed that whenever we write $[Z]_{\mathrm{vir}}$, it is to be understood with a particular choice of (X, f). When Z is smooth, a canonical choice is provided by $(X, f) = (Z, 0)$.

Remark 2.19 Let us comment on our use of the term *virtual*, which has acquired two different meanings in closely related subjects. On the one hand,

there are what are sometimes called *virtual invariants*, invariants of spaces which are additive under the scissor relation; these invariants are generally called *motivic* in this paper. Examples include the *virtual Hodge polynomial* and the *virtual Poincaré polynomial*, for which we use the terms E-polynomial and weight polynomial. On the other hand, there is the philosophy of *virtual smoothness* and the technology of *virtual cycles* for spaces such as Hilbert schemes of a threefold, which have excess dimension compared to what one would expect from deformation-obstruction theory. We use the term *virtual* exclusively in this second sense in this paper.

Remark 2.20 Let E be an object in an ind-constructible Calabi–Yau A_∞-category $\mathcal{C}$ (see [21] for the definitions of all these terms). In [21, Definition 17], Kontsevich and Soibelman associate to E a motivic weight $w(E)$. This weight lives in a certain motivic ring $\bar{\mathcal{M}}_{\mathbb{C}}^{\hat{\mu}}$, which is a completion of the ring $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ used above, quotiented by the equivalence relation, explained in [21, Sect. 4.5], which says that two motivic classes are equivalent if all cohomological realizations of these classes coincide. They claim moreover that, given a moduli space S of objects of $\mathcal{C}$, their definition gives an element in a piecewise-relative motivic ring $\bar{\mathcal{M}}_{S}^{\hat{\mu}}$.

The Kontsevich–Soibelman definition is closely related to our definition of the virtual motive of a degeneracy locus. It relies on a local description of the moduli space S as the zeros of a formal functional W built from the A_∞-structure on $\mathcal{C}$, and includes an additional factor arising from a choice of orientation data on $\mathcal{C}$, which cancels the effect of local choices. See [7, Sect. 2] for a comparison between the two approaches.

3 The Hilbert scheme of points on $\mathbb{C}^3$

3.1 Generalities on Hilbert schemes of points

For a smooth and quasi-projective variety X of dimension d, let

$$\mathrm{Sym}^n(X) = X^n/S_n$$

denote the nth symmetric product of X. For a partition α of n, let

$$\mathrm{Sym}_\alpha^n(X) \subset \mathrm{Sym}^n(X)$$

denote the locally closed subset of $\mathrm{Sym}^n(X)$ of n-tuples of points with multiplicities given by α. This gives a stratification

$$\mathrm{Sym}^n(X) = \coprod_{\alpha \vdash n} \mathrm{Sym}_\alpha^n(X).$$

Assume that the number of parts in α is $n(\alpha)$ and that α contains α_i parts of length i. Let $G_\alpha = \prod_i S_{\alpha_i}$ be the corresponding product of symmetric groups. Then there exists an open set $T_\alpha \subset X^{n(\alpha)}$ such that

$$\mathrm{Sym}^n_\alpha(X) = T_\alpha / G_\alpha$$

is a free quotient.

There is a similar story for the Hilbert scheme $\mathrm{Hilb}^n(X)$, which is stratified

$$\mathrm{Hilb}^n(X) = \coprod_{\alpha \vdash n} \mathrm{Hilb}^n_\alpha(X)$$

into locally closed strata $\mathrm{Hilb}^n_\alpha(X)$, the preimages of $\mathrm{Sym}^n_\alpha(X)$ under the Hilbert–Chow morphism. On the deepest stratum with only one part,

$$\mathrm{Hilb}^n_{(n)}(X) \to \mathrm{Sym}^n_{(n)}(X) \cong X$$

is known to be a Zariski locally trivial fibration with fiber $\mathrm{Hilb}^n(\mathbb{C}^d)_0$, the punctual Hilbert scheme of affine d-space at the origin; see e.g. [2, Corollary 4.9]. For affine space, we have a product

$$\mathrm{Hilb}^n_{(n)}(\mathbb{C}^d) \cong \mathbb{C}^d \times \mathrm{Hilb}^n(\mathbb{C}^d)_0.$$

For an arbitrary partition α, by e.g. [2, Lemma 4.10],

$$\mathrm{Hilb}^n_\alpha(X) = V_\alpha / G_\alpha$$

is a free quotient, with

$$V_\alpha = \prod_i \left(\mathrm{Hilb}^i_{(i)}(X)\right)^{\alpha_i} \setminus \widetilde{\Delta},$$

where $\widetilde{\Delta}$ denotes the locus of clusters with intersecting support. The product of Hilbert–Chow morphisms gives a map

$$V_\alpha \to \prod_i X^{\alpha_i} \setminus \Delta,$$

where as before, Δ is the big diagonal in a product of copies of X. This map is a Zariski locally trivial fibration with fiber $\prod_i (\mathrm{Hilb}^i(\mathbb{C}^d)_0)^{\alpha_i}$.

3.2 The Hilbert scheme of $\mathbb{C}^3$ as critical locus

Let T be the three-dimensional space of linear functions on $\mathbb{C}^3$, so that

$$\mathbb{C}^3 = \mathrm{Spec}\,\mathrm{Sym}^\bullet T.$$

Fix an isomorphism vol : $\wedge^3 T \cong \mathbb{C}$; this corresponds to choosing a holomorphic volume form (Calabi–Yau form) on $\mathbb{C}^3$. We start by recalling the description of the Hilbert scheme as a degeneracy locus from [35, Proof of Theorem 1.3.1].

Proposition 3.1 *The pair* (T, vol) *defines an embedding of the Hilbert scheme* $\mathrm{Hilb}^n(\mathbb{C}^3)$ *into a smooth quasi-projective variety* M_n, *which in turn is equipped with a regular function* $f_n \colon M_n \to \mathbb{C}$, *such that*

$$\mathrm{Hilb}^n(\mathbb{C}^3) = \{df_n = 0\} \subset M_n \tag{3.1}$$

is the scheme-theoretic degeneracy locus of the function f_n *on* M_n.

Proof A point $[Z] \in \mathrm{Hilb}^n(\mathbb{C}^3)$ corresponds to an embedded 0-dimensional subscheme $Z \hookrightarrow \mathbb{C}^3$ of length n, in other words to a quotient $\mathcal{O}_{\mathbb{C}^3} \to \mathcal{O}_Z$ with $H^0(\mathcal{O}_Z)$ of dimension n. Fixing an n-dimensional complex vector space V_n, the data defining a cluster consist of a linear map $T \otimes V_n \to V_n$, subject to the condition that the induced action of the tensor algebra of T factors through an action of the symmetric algebra $\mathrm{Sym}^\bullet T$, and a vector $1 \in V_n$ which generates V_n under the action.

Let

$$U_n \subset \mathrm{Hom}(T \otimes V_n, V_n) \times V_n$$

denote the space of maps with cyclic vector, the open subset where the linear span of all vectors obtained by repeated applications of the endomorphisms to the chosen vector $v \in V_n$ is the whole V_n.

Let $\chi \colon \mathrm{GL}(V_n) \to \mathbb{C}^*$ be the character given by $\chi(g) = \det(g)$. As proved in [35, Lemma 1.2.1], the open subset U_n is precisely the subset of stable points for the action of $\mathrm{GL}(V_n)$ linearized by χ. In particular, the action of $\mathrm{GL}(V_n)$ on U_n is free, and the quotient

$$M_n = \mathrm{Hom}(T \otimes V_n, V_n) \times V_n /\!/_\chi \mathrm{GL}(V_n) = U_n / \mathrm{GL}(V_n)$$

is a smooth quasi-projective GIT quotient.

Finally consider the map

$$\varphi \mapsto \mathrm{Tr}(\wedge^3 \varphi),$$

where $\wedge^3 \varphi : \bigwedge^3 T \times V_n \to V_n$ and we use the isomorphism vol before taking the trace on V_n. It is clear that this map descends to a regular map $f_n \colon M_n \to \mathbb{C}$. The equations $\{df_n = 0\}$ are just the equations which say that the action factors through the symmetric algebra; this is easy to see from the explicit description of Remark 3.2 below. Finally, as proved by [30] (in dimension 2, but the proof generalizes), the scheme cut out by these equations

is precisely the moduli scheme representing the functor of n points on $\mathbb{C}^3$. Thus, as a scheme,

$$\mathrm{Hilb}^n(\mathbb{C}^3) = \{df_n = 0\} \subset M_n. \qquad \square$$

Remark 3.2 Fixing a basis of V_n, the commutative algebra $\mathbb{C}[x, y, z]$ acts on V_n by a triple of matrices A, B, C. The variety M_n is the space of triples with generating vector, where the matrices do not necessarily commute, modulo the action of $GL(V_n)$. The map f_n on triples of matrices is given by

$$(A, B, C) \mapsto \mathrm{Tr}[A, B]C.$$

Written explicitly in terms of the matrix entries,

$$\mathrm{Tr}[A, B]C = \sum_{i,k} \sum_{j} (A_{ij} B_{jk} - B_{ij} A_{jk}) C_{ki},$$

and so

$$\partial_{C_{ki}} \mathrm{Tr}[A, B]C = \sum_{j} (A_{ij} B_{jk} - B_{ij} A_{jk}) = 0$$

for all i, k indeed means that A and B commute.

Remark 3.3 This description of $\mathrm{Hilb}^n(\mathbb{C}^3)$ can also be written in the language of quivers. Consider the quiver consisting of two nodes, with a single arrow from the first node to the second (corresponding to v) and three additional loops on the second node (corresponding to A, B, and C), with relations coming from the super-potential $W = A[B, C]$. Then the space M_n can be identified with stable representations of this bound quiver, with dimension vector $(1, n)$, and specific choice of stability parameter which matches the GIT stability condition described above.

3.3 Some properties of the family

Lemma 3.4 *There exists a linearized* $T = (\mathbb{C}^*)^3$*-action on* M_n *such that* $f_n \colon M_n \to \mathbb{C}$ *is equivariant with respect to the primitive character* $\chi \colon T \to \mathbb{C}^*$ *given by* $\chi(t_1, t_2, t_3) = t_1 t_2 t_3$. *Moreover, the action of the diagonal 1-parameter subgroup is circle compact.*

Proof In the notation of Remark 3.2, consider the T-action

$$(t_1, t_2, t_3) \circ (A, B, C, v) = (t_1 A, t_2 B, t_3 C, t_1 t_2 t_3 v)$$

on the space

$$U_n \subset \mathrm{Hom}(V_n, V_n)^{\times 3} \times V_n$$

of maps with cyclic vector. The map $(A, B, C, v) \mapsto \mathrm{Tr}[A, B]C$ is T-equivariant with respect to the character $\chi(t_1, t_2, t_3) = t_1 t_2 t_3$. Moreover, the T-action on U_n commutes with the $\mathrm{GL}(V_n)$-action acting freely on U_n, so descends to the quotient M_n. The T-action on U_n lifts to the linearization and hence defines a linearized action of T on M_n.

Consider the $\mathbb{C}^*$-action on $\mathrm{Hom}(V_n, V_n)^{\times 3} \times V_n$ induced by the diagonal subgroup in T. Let

$$M_n^0 = \mathrm{Hom}(V_n, V_n)^{\times 3} \times V_n /\!/_0 \, \mathrm{GL}(V_n)$$

be the affine quotient, the GIT quotient at the trivial character. Then by general GIT, there is a natural proper map $\pi_n \colon M_n \to M_n^0$, which is $\mathbb{C}^*$-equivariant. On the other hand, it is clear that the only $\mathbb{C}^*$-fixed point in M_n^0 is the image of the origin in $\mathrm{Hom}(V_n, V_n)^{\times 3} \times V_n$, and all $\mathbb{C}^*$-orbits in M_n^0 have this point in their closure as $\lambda \to 0$. By the properness of π_n, the $\mathbb{C}^*$-fixed points in M_n form a complete subvariety, and all limits as $\lambda \to 0$ exist. Thus the diagonal $\mathbb{C}^*$-action on M_n is circle compact. $\qquad\square$

The function $f_n : M_n \to \mathbb{C}$ is not proper, so we cannot apply Proposition 2.11, but as a corollary to the above lemma, we may apply Proposition 2.12 instead.

Corollary 3.5 *For each n, the absolute motivic vanishing cycle of the family $f_n \colon M_n \to \mathbb{C}$ can be computed as the motivic difference*

$$[\varphi_{f_n}] = \left[f_n^{-1}(1)\right] - \left[f_n^{-1}(0)\right] \in \mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}.$$

3.4 The virtual motive of the Hilbert scheme

As a consequence of Proposition 3.1, the singular space $\mathrm{Hilb}^n(\mathbb{C}^3)$ acquires relative and absolute virtual motives $[\mathrm{Hilb}^n(\mathbb{C}^3)]_{\mathrm{relvir}}$ and $[\mathrm{Hilb}^n(\mathbb{C}^3)]_{\mathrm{vir}}$. Let us stress that, a priori, these classes depend on the chosen linear Calabi–Yau structure on $\mathbb{C}^3$; it is indeed not clear to us what happens to the relative motive if we apply an automorphism of $\mathrm{Hilb}^n(\mathbb{C}^3)$ induced by a non-linear regular automorphism of $\mathbb{C}^3$.

Define the motivic Donaldson–Thomas partition function of $\mathbb{C}^3$ to be

$$Z_{\mathbb{C}^3}(t) = \sum_{n=0}^{\infty} \left[\mathrm{Hilb}^n(\mathbb{C}^3)\right]_{\mathrm{vir}} t^n \in \mathcal{M}_{\mathbb{C}}[\![t]\!].$$

By Proposition 2.16,

$$W\left(\left[\mathrm{Hilb}^n(\mathbb{C}^3)\right]_{\mathrm{vir}}, q^{\frac{1}{2}} = -1\right) \in \mathbb{Z}$$

is the Donaldson–Thomas invariant, the signed number of 3-dimensional partitions of n (see [2, §4.1]). Hence the Euler characteristic specialization

$$\chi Z_{\mathbb{C}^3}(t) = M(-t)$$

is the signed MacMahon function.

Using the stratification

$$\mathrm{Hilb}^n(\mathbb{C}^3) = \coprod_{\alpha \vdash n} \mathrm{Hilb}^n_\alpha(\mathbb{C}^3),$$

the relative virtual motive

$$\left[\mathrm{Hilb}^n(\mathbb{C}^3)\right]_{\mathrm{relvir}} \in \mathcal{M}^{\hat{\mu}}_{\mathrm{Hilb}^n(\mathbb{C}^3)}$$

can be restricted to define the relative virtual motives

$$\left[\mathrm{Hilb}^n_\alpha(\mathbb{C}^3)\right]_{\mathrm{relvir}} \in \mathcal{M}^{\hat{\mu}}_{\mathrm{Hilb}^n_\alpha(\mathbb{C}^3)}$$

for the strata. We additionally define the relative virtual motive of the punctual Hilbert scheme

$$\left[\mathrm{Hilb}^n(\mathbb{C}^3)_0\right]_{\mathrm{relvir}} \in \mathcal{M}^{\hat{\mu}}_{\mathrm{Hilb}^n(\mathbb{C}^3)_0}$$

by restricting $[\mathrm{Hilb}^n_{(n)}(\mathbb{C}^3)]_{\mathrm{relvir}}$ to

$$\{0\} \times \mathrm{Hilb}^n(\mathbb{C}^3)_0 \subset \mathbb{C}^3 \times \mathrm{Hilb}^n(\mathbb{C}^3)_0 \cong \mathrm{Hilb}^n_{(n)}(\mathbb{C}^3).$$

Associated to each relative virtual motive, we have the absolute motives

$$\left[\mathrm{Hilb}^n_\alpha(\mathbb{C}^3)\right]_{\mathrm{vir}}, \left[\mathrm{Hilb}^n(\mathbb{C}^3)_0\right]_{\mathrm{vir}} \in \mathcal{M}^{\hat{\mu}}_{\mathbb{C}},$$

and the absolute motives satisfy

$$\left[\mathrm{Hilb}^n(\mathbb{C}^3)\right]_{\mathrm{vir}} = \sum_{\alpha \vdash n} \left[\mathrm{Hilb}^n_\alpha(\mathbb{C}^3)\right]_{\mathrm{vir}}.$$

We now collect some properties of these virtual motives.

Proposition 3.6

1. *The absolute virtual motives* $[\mathrm{Hilb}^n_\alpha(\mathbb{C}^3)]_{\mathrm{vir}}$ *and* $[\mathrm{Hilb}^n(\mathbb{C}^3)_0]_{\mathrm{vir}}$ *live in the subring* $\mathcal{M}_{\mathbb{C}} \subset \mathcal{M}^{\hat{\mu}}_{\mathbb{C}}$.
2. *On the closed stratum,*

$$\left[\mathrm{Hilb}^n_{(n)}(\mathbb{C}^3)\right]_{\mathrm{vir}} = \mathbb{L}^3 \cdot \left[\mathrm{Hilb}^n(\mathbb{C}^3)_0\right]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}}.$$

3. *More generally, for a general stratum,*

$$\left[\operatorname{Hilb}_\alpha^n(\mathbb{C}^3)\right]_{\mathrm{vir}} = \pi_{G_\alpha}\left(\left[\prod_i (\mathbb{C}^3)^{\alpha_i} \setminus \Delta\right] \cdot \prod_i \left[\operatorname{Hilb}^i(\mathbb{C}^3)_0^{\alpha_i}\right]_{\mathrm{vir}}\right),$$

where π_{G_α} denotes the quotient map (2.5).

Proof We start by proving (1) and (2) together. On the one hand, consider the closed stratum

$$\operatorname{Hilb}_{(n)}^n(\mathbb{C}^3) \cong \mathbb{C}^3 \times \operatorname{Hilb}^n(\mathbb{C}^3)_0,$$

with projections p_i to the factors. By the invariance of the construction under the translation action of $\mathbb{C}^3$ on itself, the relative virtual motive is

$$\left[\operatorname{Hilb}_{(n)}^n(\mathbb{C}^3)\right]_{\mathrm{relvir}} = p_2^*\left[\operatorname{Hilb}^n(\mathbb{C}^3)_0\right]_{\mathrm{relvir}}.$$

Taking absolute motives,

$$\left[\operatorname{Hilb}_{(n)}^n(\mathbb{C}^3)\right]_{\mathrm{vir}} = \mathbb{L}^3 \cdot \left[\operatorname{Hilb}^n(\mathbb{C}^3)_0\right]_{\mathrm{vir}}, \tag{3.2}$$

with both sides living a priori in $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$.

On the other hand, as it is well known, the affinization of the Hilbert scheme $\operatorname{Hilb}^n(\mathbb{C}^3)$ is the symmetric product. The conditions of Proposition 2.13, hold, since the cubic hypersurface given by the function f_n is reduced. Applying Proposition 2.13, we see that the relative virtual motives on the strata of the Hilbert scheme have trivial $\hat{\mu}$-action over the corresponding strata in the symmetric product. Hence the absolute motives $[\operatorname{Hilb}_\alpha^n(\mathbb{C}^3)]_{\mathrm{vir}}$ also carry trivial $\hat{\mu}$-action. The same statement for the punctual Hilbert scheme then follows from (3.2), with (3.2) holding in fact in $\mathcal{M}_{\mathbb{C}}$.

To prove (3), consider the diagram

$$
\begin{array}{ccccc}
V_\alpha & \hookrightarrow & W_\alpha & \hookrightarrow & \prod_i \operatorname{Hilb}^i(\mathbb{C}^3)^{\alpha_i} \\
\downarrow{\scriptstyle q_\alpha} & & \downarrow & & \\
\operatorname{Hilb}_\alpha^n(\mathbb{C}^3) & \hookrightarrow & U_\alpha & \hookrightarrow & \operatorname{Hilb}^n(\mathbb{C}^3)
\end{array}
$$

from [2, Lemma 4.10]. Here W_α is the locus of points in $\prod_i \operatorname{Hilb}^i(\mathbb{C}^3)^{\alpha_i}$ which parametrizes subschemes with disjoint support, U_α is the image of W_α in $\operatorname{Hilb}^n(\mathbb{C}^3)$, and V_α makes the left hand square Cartesian. The first vertical map q_α is Galois whereas the second one is étale. The first inclusion in each row is closed whereas the second one is open.

Consider the construction of the Hilbert scheme, as a space of commuting matrices with cyclic vector, in a neighborhood of U_α in the space of matrices. Pulling back to the cover V_α, we see that a point of V_α is represented by tuples of commuting matrices X_j, Y_j, Z_j acting on some linear spaces V_j, with generating vectors v_j. The covering map is simply obtained by direct sum: $V = \bigoplus V_j$ acted on by $X = \bigoplus X_j$ and Y, Z defined similarly. The vector $v = \bigoplus v_j$ is cyclic for X, Y, Z exactly because the eigenvalues of the X_j, Y_j, Z_j do not all coincide for different j; this is the disjoint support property of points of V_α.

On the other hand, clearly

$$\operatorname{Tr} X[Y, Z] = \sum_j \operatorname{Tr} X_j[Y_j, Z_j]$$

for block-diagonal matrices. Thus, the Thom–Sebastiani Theorem 2.9 implies that the pullback relative motive

$$q_\alpha^* \big[\operatorname{Hilb}_\alpha^n(\mathbb{C}^3)\big]_{\text{relvir}} \in \mathcal{M}_{V_\alpha}^{\hat\mu}$$

is equal to the restriction to V_α of the $\star$-products of the relative virtual motives of the punctual Hilbert schemes $\operatorname{Hilb}_{(i)}^i(\mathbb{C}^3)$. Taking the associated absolute motives, using the locally trivial fibration on V_α along with (2), we get

$$q_\alpha^* \big[\operatorname{Hilb}_\alpha^n(\mathbb{C}^3)\big]_{\text{vir}} = \Big[\prod_i (\mathbb{C}^3)^{\alpha_i} \setminus \Delta\Big] \cdot \prod_i \big[\operatorname{Hilb}^i(\mathbb{C}^3)_0^{\alpha_i}\big]_{\text{vir}} \in \mathcal{M}_{\mathbb{C}}.$$

Here, using (1), the $\star$-product became the ordinary product. By Lemma 2.4, the G_α-action extends to this class, and (3) follows. $\qquad \square$

3.5 Computing the motivic partition function of $\mathbb{C}^3$

The core result of this paper is the computation of

$$Z_{\mathbb{C}^3}(t) = \sum_{n=0}^\infty \big[\operatorname{Hilb}^n(\mathbb{C}^3)\big]_{\text{vir}} t^n,$$

the motivic Donaldson–Thomas partition function of $\mathbb{C}^3$.

Theorem 3.7 *The motivic partition function $Z_{\mathbb{C}^3}(t)$ lies in $\mathcal{M}_{\mathbb{C}}[[t]] \subset \mathcal{M}_{\mathbb{C}}^{\hat\mu}[[t]]$ and is given by*

$$Z_{\mathbb{C}^3}(t) = \prod_{m=1}^\infty \prod_{k=0}^{m-1} \big(1 - \mathbb{L}^{k+2-m/2} t^m\big)^{-1}. \tag{3.3}$$

Proof Recall that

$$\mathrm{Hilb}^n(\mathbb{C}^3) = \{df_n = 0\},$$

where f_n is the function

$$f_n(A, B, C, v) = \mathrm{Tr}\, A[B, C]$$

defined on the smooth variety

$$M_n = U_n / \mathrm{GL}(V_n),$$

where V_n is an n-dimensional vector space, and

$$U_n \subset \mathrm{Hom}(V_n, V_n)^3 \times V_n$$

is the open set of points (A, B, C, v) satisfying the stability condition that monomials in A, B, C applied to v generate V_n.

By Corollary 3.5, to compute the virtual motive, we need to compute the motivic difference of the fibers $f_n^{-1}(1)$ and $f_n^{-1}(0)$.

Let

$$Y_n = \left\{(A, B, C, v) : \mathrm{Tr}\, A[B, C] = 0\right\} \subset \mathrm{Hom}(V_n, V_n)^3 \times V_n,$$

and let

$$Z_n = \left\{(A, B, C, v) : \mathrm{Tr}\, A[B, C] = 1\right\} \subset \mathrm{Hom}(V_n, V_n)^3 \times V_n.$$

The isomorphism

$$\mathrm{Hom}(V_n, V_n)^3 \times V_n \setminus Y_n \cong \mathbb{C}^* \times Z_n$$

given by

$$(A, B, C, v) \mapsto \left(\mathrm{Tr}\, A[B, C], \left(\mathrm{Tr}\, A[B, C]\right)^{-1} A, B, C, v\right)$$

yields the motivic relation

$$[Y_n] + (\mathbb{L} - 1)[Z_n] = \left[\mathrm{Hom}(V_n, V_n) \times V_n\right] = \mathbb{L}^{3n^2 + n},$$

equivalently

$$(1 - \mathbb{L})\left([Y_n] - [Z_n]\right) = \mathbb{L}^{3n^2 + n} - \mathbb{L}[Y_n].$$

The space Y_n stratifies as a union

$$Y_n = Y'_n \sqcup Y''_n$$

where Y_n' consists of the locus where B and C commute and Y_n'' is its complement. Projections onto the B and C factors induce maps

$$Y_n' \to C_n, \qquad Y_n'' \to \left\{ \mathbb{C}^{2n^2} \setminus C_n \right\}$$

where $C_n \subset \mathbb{C}^{2n^2}$ is the commuting variety. The first map is projection on the second factor under the product decomposition $Y_n' \cong \mathbb{C}^{n^2+n} \times C_n$ and the second map is a Zariski trivial fibration with fibers isomorphic to $\mathbb{C}^{n^2-1+n}$. Indeed, for fixed B and C with $[B, C] \neq 0$, the condition $\operatorname{Tr} A[B, C] = 0$ is a single non-trivial linear condition on the matrices A. Moreover, the fibration is Zariski trivial over the open cover whose sets are given by the condition that some given matrix entry of $[B, C]$ is non-zero. Thus the above stratification yields the equation of motives

$$[Y_n] = \mathbb{L}^{n^2+n}[C_n] + \mathbb{L}^{n^2-1+n} \left(\mathbb{L}^{2n^2} - [C_n] \right).$$

Substituting into the previous equation and canceling terms we obtain

$$(1 - \mathbb{L}) \left([Y_n] - [Z_n] \right) = -\mathbb{L}^{n^2+n} \left(\mathbb{L}[C_n] - [C_n] \right).$$

Writing

$$w_n = [Y_n] - [Z_n],$$

we get the basic equation

$$w_n = \mathbb{L}^{n(n+1)}[C_n]. \tag{3.4}$$

We now need to incorporate the stability condition. We call the smallest subspace of V_n containing v and invariant under the action of A, B, and C the (A, B, C)-*span of* v. Let

$$X_n^k = \left\{ (A, B, C, v) : \text{the } (A, B, C)\text{-span of } v \text{ has dimension } k \right\}$$

and let

$$Y_n^k = Y_n \cap X_n^k, \qquad Z_n^k = Z_n \cap X_n^k.$$

We compute the motive of Y_n^k as follows. There is a Zariski locally trivial fibration

$$Y_n^k \to Gr(k, n)$$

given by sending (A, B, C, v) to the (A, B, C)-span of v.

To compute the motive of the fiber of this map, we choose a basis of V_n so that the first k vectors are in the (A, B, C)-span of v. In this basis, (A, B, C, v) in a fixed fiber all have the form

$$A = \begin{pmatrix} A_0 & A' \\ 0 & A_1 \end{pmatrix}, \qquad B = \begin{pmatrix} B_0 & B' \\ 0 & B_1 \end{pmatrix}, \qquad C = \begin{pmatrix} C_0 & C' \\ 0 & C_1 \end{pmatrix},$$

$$v = \begin{pmatrix} v_0 \\ 0 \end{pmatrix}$$

where (A_0, B_0, C_0) are $k \times k$ matrices, (A', B', C') are $k \times (n-k)$ matrices, (A_1, B_1, C_1) are $(n-k) \times (n-k)$ matrices, and v_0 is a k-vector.

Thus a fiber of $Y_n^k \to Gr(k, n)$ is given by the locus of

$$\{(A_0, B_0, C_0, v_0), (A_1, B_1, C_1), (A', B', C')\}$$

satisfying

$$\operatorname{Tr} A[B, C] = \operatorname{Tr} A_0[B_0, C_0] + \operatorname{Tr} A_1[B_1, C_1] = 0.$$

This space splits into a factor $\mathbb{C}^{3(n-k)k}$, corresponding to the triple (A', B', C'), and a remaining factor which stratifies into a union of

$$\{\operatorname{Tr} A_0[B_0, C_0] = \operatorname{Tr} A_1[B_1, C_1] = 0\}$$

and

$$\{\operatorname{Tr} A_0[B_0, C_0] = - \operatorname{Tr} A_1[B_1, C_1] \neq 0\}.$$

Projection on the (A_0, B_0, C_0, v_0) and (A_1, B_1, C_1) factors induces a product structure on the above strata so that the corresponding motives are given by

$$[Y_k^k] \cdot [Y_{n-k}] \mathbb{L}^{-(n-k)}$$

and

$$(\mathbb{L} - 1)[Z_k^k][Z_{n-k}] \mathbb{L}^{-(n-k)}$$

respectively. Putting this all together yields

$$[Y_n^k] = \mathbb{L}^{3(n-k)k} \begin{bmatrix} n \\ k \end{bmatrix}_{\mathbb{L}} \left([Y_k^k] \cdot [Y_{n-k}] \cdot \mathbb{L}^{-(n-k)} \right.$$

$$\left. + (\mathbb{L} - 1) \cdot [Z_k^k] \cdot [Z_{n-k}] \cdot \mathbb{L}^{-(n-k)} \right).$$

A similar analysis yields

$$[Z_n^k] = \mathbb{L}^{3(n-k)k} \begin{bmatrix} n \\ k \end{bmatrix}_{\mathbb{L}} \left([Y_k^k] \cdot [Z_{n-k}] \cdot \mathbb{L}^{-(n-k)} \right.$$

$$\left. + (\mathbb{L} - 2) \cdot [Z_k^k] \cdot [Z_{n-k}] \cdot \mathbb{L}^{-(n-k)} + [Z_k^k] \cdot [Y_{n-k}] \cdot \mathbb{L}^{-(n-k)} \right).$$

We are interested in the difference

$$
\begin{aligned}
w_n^k &= [Y_n^k] - [Z_n^k] \\
&= \mathbb{L}^{(3k-1)(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{\mathbb{L}} \left(w_{n-k}[Y_k^k] - w_{n-k}[Z_k^k] \right) \\
&= \mathbb{L}^{(n-k)(n+2k)} \begin{bmatrix} n \\ k \end{bmatrix}_{\mathbb{L}} [C_{n-k}] w_k^k,
\end{aligned}
$$

where we used (3.4) for the last equality.

Observing that $Y_n = \bigsqcup_{k=0}^n Y_n^k$ and $Z_n = \bigsqcup_{k=0}^n Z_n^k$, we get

$$
w_n^n = w_n - \sum_{k=0}^{n-1} w_n^k,
$$

into which we substitute our equations for w_n and w_n^k to derive the following recursion for w_n^n:

$$
w_n^n = \mathbb{L}^{n(n+1)}[C_n] - \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_{\mathbb{L}} \mathbb{L}^{(n-k)(n+2k)}[C_{n-k}] w_k^k. \tag{3.5}
$$

We can now compute the virtual motive of the Hilbert scheme. By Proposition 2.12, we get

$$
[\varphi_{f_n}] = -[f_n^{-1}(0)] + [f_n^{-1}(1)] = -\frac{[Y_n^n]}{[\mathrm{GL}_n(\mathbb{C})]} + \frac{[X_n^n]}{[\mathrm{GL}_n(\mathbb{C})]} = -\frac{w_n^n}{\mathbb{L}^{\binom{n}{2}}[n]_{\mathbb{L}}!}.
$$

The dimension of M_n is $2n^2 + n$, so we find

$$
\left[\mathrm{Hilb}^n(\mathbb{C}^3) \right]_{\mathrm{vir}} = -\mathbb{L}^{-n^2 - n/2}[\varphi_{f_n}] = \mathbb{L}^{-\frac{3n^2}{2}} \frac{w_n^n}{[n]_{\mathbb{L}}!}.
$$

Working in the ring $\mathcal{M}_{\mathbb{C}}[(1 - \mathbb{L}^n)^{-1} : n \geq 1]$, we divide (3.5) by $\mathbb{L}^{3n^2/2}[n]_{\mathbb{L}}!$ and rearrange to obtain

$$
\widetilde{c}_n \mathbb{L}^{n/2} = \sum_{k=0}^n \widetilde{c}_{n-k} \left[\mathrm{Hilb}^k(\mathbb{C}^3) \right]_{\mathrm{vir}} \mathbb{L}^{-(n-k)/2},
$$

where

$$
\widetilde{c}_n = \mathbb{L}^{-\binom{n}{2}} \frac{C_n}{[n]_{\mathbb{L}}!}.
$$

is the renormalized motive (2.1) of the space C_n of commuting pairs of matrices. Multiplying by t^n and summing, we get

$$C\big(t\mathbb{L}^{1/2}\big) = Z_{\mathbb{C}^3}(t)C\big(t\mathbb{L}^{-1/2}\big),$$

with $C(t)$ as in (2.2). Thus using Proposition 2.1 (cf. Remark 2.2) we obtain

$$Z_{\mathbb{C}^3}(t) = \frac{C(t\mathbb{L}^{1/2})}{C(t\mathbb{L}^{-1/2})} = \prod_{m=1}^{\infty} \prod_{j=0}^{\infty} \frac{(1 - \mathbb{L}^{1-j+m/2}t^m)^{-1}}{(1 - \mathbb{L}^{1-j-m/2}t^m)^{-1}}$$

$$= \prod_{m=1}^{\infty} \prod_{j=0}^{m-1} \big(1 - \mathbb{L}^{1-j+m/2}t^m\big)^{-1} = \prod_{m=1}^{\infty} \prod_{k=0}^{m-1} \big(1 - \mathbb{L}^{2+k-m/2}t^m\big)^{-1}$$

$$\tag{3.6}$$

which completes the proof of Theorem 3.7.   □

Remark 3.8 Some formulae in the above proof appear also in recent work of Reineke [32] and Kontsevich–Soibelman [22]. In particular, the twisted quotient (3.6) appears in [32, Proposition 3.3]. The twisted quotient is applied later in [32, Sect. 4] to a generating series of stacky quotients, analogously to our series C defined in (2.1). Reineke's setup is more general, dealing with arbitrary quivers, but also more special, since there are no relations.

Remark 3.9 The result of Theorem 3.7 shows in particular that the absolute virtual motives of $\mathrm{Hilb}^n(\mathbb{C}^3)$ are independent of the chosen linear Calabi–Yau structure on $\mathbb{C}^3$.

Remark 3.10 The first non-trivial example is the case of four points, with $\mathrm{Hilb}^4(\mathbb{C}^3)$ irreducible and reduced but singular. The E-polynomial realization of the virtual motive on $\mathrm{Hilb}^4(\mathbb{C}^3)$ was computed earlier by [12]. The result, up to the different normalization used there, coincides with the $t = 4$ term of the result above.

Remark 3.11 The Euler characteristic specialization of our formula is obtained by setting $\mathbb{L}^{\frac{1}{2}} = -1$. This immediately leads to

$$\chi Z_{\mathbb{C}^3}(t) = \prod_{m=1}^{\infty} \big(1 - (-t)^m\big)^{-m} = M(-t),$$

where $M(t)$ is the MacMahon function enumerating 3D partitions. The standard proof of

$$\chi Z_{\mathbb{C}^3}(t) = M(-t)$$

is by torus localization [2, 27]. Our argument gives a new proof of this result, which is independent of the combinatorics of 3-dimensional partitions. Indeed, by combining the two arguments, we obtain a new (albeit non-elementary) proof of MacMahon's formula. It is of course conceivable that Theorem 3.7 also has a proof by torus localization (perhaps after the E-polynomial specialization). But as the computations of [12] show, this has to be nontrivial, since the fixed point contributions are not pure weight.

4 The Hilbert scheme of points of a general threefold

4.1 The virtual motive of the Hilbert scheme

Let X be a smooth and quasi-projective threefold. Recall the stratification of $\mathrm{Hilb}^n(X)$ by strata $\mathrm{Hilb}^n_\alpha(X)$ indexed by partitions α of n. Proposition 3.6 dictates the following recipe for associating a virtual motive to the Hilbert scheme and its strata.

Definition 4.1 We define virtual motives

$$\left[\mathrm{Hilb}^n_\alpha(X)\right]_{\mathrm{vir}} \in \mathcal{M}_\mathbb{C} \quad \text{and} \quad \left[\mathrm{Hilb}^n(X)\right]_{\mathrm{vir}} \in \mathcal{M}_\mathbb{C}$$

as follows.

1. On the deepest stratum,

$$\left[\mathrm{Hilb}^n_{(n)}(X)\right]_{\mathrm{vir}} = [X] \cdot \left[\mathrm{Hilb}^n(\mathbb{C}^3)_0\right]_{\mathrm{vir}},$$

   where $[\mathrm{Hilb}^n(\mathbb{C}^3)_0]_{\mathrm{vir}}$ is as defined in Sect. 3.4.
2. More generally, on all strata,

$$\left[\mathrm{Hilb}^n_\alpha(X)\right]_{\mathrm{vir}} = \pi_{G_\alpha}\left(\left[\prod_i X^{\alpha_i} \setminus \Delta\right] \cdot \prod_i \left[\mathrm{Hilb}^i(\mathbb{C}^3)_0^{\alpha_i}\right]_{\mathrm{vir}}\right),$$

   where the motivic classes $[\prod_i X^{\alpha_i} \setminus \Delta]$ and $\prod_i[\mathrm{Hilb}^i(\mathbb{C}^d)_0]^{\alpha_i}_{\mathrm{vir}}$ carry G_α-actions, and π_{G_α} denotes the quotient map (2.5).
3. Finally

$$\left[\mathrm{Hilb}^n(X)\right]_{\mathrm{vir}} = \sum_\alpha \left[\mathrm{Hilb}^n_\alpha(X)\right]_{\mathrm{vir}}.$$

Of course by Proposition 3.6, this definition reconstructs the virtual motives of $\mathrm{Hilb}^n(\mathbb{C}^3)$ from those of the punctual Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C}^3)_0$ consistently with its original definition.

4.2 The partition function of the Hilbert scheme

Let

$$Z_X(t) = \sum_{n=0}^{\infty} \left[\mathrm{Hilb}^n(X)\right]_{\mathrm{vir}} t^n \in \mathcal{M}_{\mathbb{C}}[[t]]$$

be the motivic degree zero Donaldson–Thomas partition function of a smooth quasi-projective threefold X. We will derive expressions for this series and its specializations from Theorem 3.7. We use the Exp map and power structure on the ring of motivic weights introduced in Sect. 2.5 throughout this section.

Let

$$Z_{\mathbb{C}^3,0}(t) = \sum_{n=0}^{\infty} \left[\mathrm{Hilb}^n(\mathbb{C}^3)_0\right]_{\mathrm{vir}} t^n$$

be the generating series of virtual motives of the punctual Hilbert schemes of $\mathbb{C}^3$ at the origin. The following statement is the virtual motivic analogue of Cheah's [6, Main Theorem].

Proposition 4.2 *We have*

$$Z_X(t) = Z_{\mathbb{C}^3,0}(t)^{[X]}.$$

Proof We have

$$Z_X(t) = 1 + \sum_{\alpha} \left[\mathrm{Hilb}^n_{\alpha}(X)\right]_{\mathrm{vir}} t^{|\alpha|}$$

$$= 1 + \sum_{\alpha} \pi_{G_{\alpha}} \left(\left[\prod_i X^{\alpha_i} \setminus \Delta\right] \cdot \prod_i \left[\mathrm{Hilb}^i(\mathbb{C}^d)_0\right]^{\alpha_i}_{\mathrm{vir}} \right) t^{|\alpha|}$$

$$= \left(1 + \sum_{n \geq 1} \left[\mathrm{Hilb}^n(\mathbb{C}^d)_0\right]_{\mathrm{vir}} t^n \right)^{[X]}.$$

Here, first we use Definition 4.1(3), then Definition 4.1(2), and finally the power structure formula (2.6). $\qquad\square$

Theorem 4.3 *Let X be a smooth and quasi-projective threefold. Then*

$$Z_X(-t) = \mathrm{Exp}\left(\frac{-t[X]_{\mathrm{vir}}}{(1 + \mathbb{L}^{\frac{1}{2}}t)(1 + \mathbb{L}^{-\frac{1}{2}}t)} \right). \tag{4.1}$$

Proof We begin by writing the formula from Theorem 3.7 using the power structure on $\mathcal{M}_{\mathbb{C}}$ and the Exp function defined in Sect. 2.5.

$$
\begin{aligned}
Z_{\mathbb{C}^3}(-t) &= \prod_{m=1}^{\infty} \prod_{k=0}^{m-1} \left(1 - \mathbb{L}^{2+k-\frac{m}{2}}(-t)^m\right)^{-1} \\
&= \prod_{m=1}^{\infty} \prod_{k=0}^{m-1} \left(1 - \left(-\mathbb{L}^{\frac{1}{2}}\right)^{4+2k-m} t^m\right)^{-1} \\
&= \prod_{m=1}^{\infty} \left(1 - t^m\right)^{-\sum_{k=0}^{m-1}\left(-\mathbb{L}^{\frac{1}{2}}\right)^{4+2k-m}} \\
&= \mathrm{Exp}\left(\sum_{m=1}^{\infty} t^m \sum_{k=0}^{m-1} \left(-\mathbb{L}^{\frac{1}{2}}\right)^{4+2k-m}\right) \\
&= \mathrm{Exp}\left(\sum_{m=1}^{\infty} t^m \left(-\mathbb{L}^{\frac{1}{2}}\right)^{4-m} \cdot \frac{1 - \left(-\mathbb{L}^{\frac{1}{2}}\right)^{2m}}{1 - \mathbb{L}}\right) \\
&= \mathrm{Exp}\left(\frac{\mathbb{L}^2}{1 - \mathbb{L}} \sum_{m=1}^{\infty} \left(\left(-\mathbb{L}^{-\frac{1}{2}}t\right)^m - \left(-\mathbb{L}^{\frac{1}{2}}t\right)^m\right)\right) \\
&= \mathrm{Exp}\left(\frac{\mathbb{L}^{\frac{3}{2}}}{\mathbb{L}^{-\frac{1}{2}} - \mathbb{L}^{\frac{1}{2}}} \cdot \left(\frac{-\mathbb{L}^{-\frac{1}{2}}t}{1 + \mathbb{L}^{-\frac{1}{2}}t} - \frac{-\mathbb{L}^{\frac{1}{2}}t}{1 + \mathbb{L}^{\frac{1}{2}}t}\right)\right) \\
&= \mathrm{Exp}\left(\frac{-\mathbb{L}^{\frac{3}{2}}t}{(1 + \mathbb{L}^{-\frac{1}{2}}t)(1 + \mathbb{L}^{\frac{1}{2}}t)}\right).
\end{aligned}
$$

Replacing t by $-t$ in Proposition 4.2, and letting $X = \mathbb{C}^3$, we find that

$$
Z_{\mathbb{C}^3,0}(-t) = \mathrm{Exp}\left(\frac{-\mathbb{L}^{-\frac{3}{2}}t}{(1 + \mathbb{L}^{-\frac{1}{2}}t)(1 + \mathbb{L}^{\frac{1}{2}}t)}\right).
$$

Another application of Proposition 4.2 concludes the proof.   □

Our formula fits nicely with the corresponding formulas for surfaces, curves, and points. Göttsche's formula [15, 16] for a smooth quasi-projective surface S, rewritten in motivic exponential form in [17, Statement 4], reads

$$
\sum_{n=0}^{\infty} \left[\mathrm{Hilb}^n(S)\right] t^n = \mathrm{Exp}\left(\frac{[S]t}{1 - \mathbb{L}t}\right) \tag{4.2}
$$

and for a smooth curve C we have

$$\sum_{n=0}^{\infty}\big[\mathrm{Hilb}^n(C)\big]t^n = \sum_{n=0}^{\infty}\big[\mathrm{Sym}^n(C)\big]t^n = \mathrm{Exp}\big([C]t\big).$$

Finally, for completeness, consider P, a collection of N points. Then

$$\sum_{n=0}^{\infty}\big[\mathrm{Hilb}^n(P)\big]t^n = \sum_{n=0}^{N}\binom{N}{n}t^n = \frac{(1-t^2)^N}{(1-t)^N} = \mathrm{Exp}\big([P]t(1-t)\big).$$

Since $\mathrm{Hilb}^n(X)$ is smooth and of the expected dimension when the dimension of X is 0, 1, or 2, the virtual motives are given by (2.11):

$$\big[\mathrm{Hilb}^n(X)\big]_{\mathrm{vir}} = \mathbb{L}^{-\frac{n\dim X}{2}}\big[\mathrm{Hilb}^n(X)\big].$$

The series

$$Z_X(t) = \sum_{n=0}^{\infty}\big[\mathrm{Hilb}^n(X)\big]_{\mathrm{vir}}t^n$$

is thus well defined for any X of dimension 0, 1, 2, or 3. For $\dim X \geq 4$, $Z_X(t)$ is defined to order t^3 since $\mathrm{Hilb}^n(X)$ is smooth for $n \leq 3$ in all dimensions. In order to write $Z_X(t)$ as a motivic exponential, we must introduce a sign. Let $T = (-1)^d t$ where $d = \dim X$. Then for d equal to 0, 1, or 2, we have

$$Z_X(T) = \sum_{n=0}^{\infty}\big[\mathrm{Hilb}^n(X)\big]\mathbb{L}^{-\frac{dn}{2}}T^n = \sum_{n=0}^{\infty}\big[\mathrm{Hilb}^n(X)\big]\big((-\mathbb{L}^{\frac{1}{2}})^{-d}t\big)^n.$$

Applying the substitution rule (2.8) to the above formulas and including the $d = 3$ case from Theorem 4.3, we find

$$Z_X(T) = \mathrm{Exp}\big(T[X]_{\mathrm{vir}}G_d(T)\big),$$

where

$$G_d(T) = \begin{cases} 1-T & d=0 \\ 1 & d=1 \\ (1-T)^{-1} & d=2 \\ (1-\mathbb{L}^{\frac{1}{2}}T)^{-1}(1-\mathbb{L}^{-\frac{1}{2}}T)^{-1} & d=3. \end{cases}$$

The above can be written uniformly as

$$G_d(T) = \mathrm{Exp}\big(T\big[\mathbb{P}^{d-2}\big]_{\mathrm{vir}}\big),$$

where we have defined $[\mathbb{P}^N]_{\text{vir}}$ for negative N via the equation

$$\left[\mathbb{P}^N\right]_{\text{vir}} = \mathbb{L}^{-\frac{N}{2}} \cdot \frac{\mathbb{L}^{N+1} - 1}{\mathbb{L} - 1}. \tag{4.3}$$

In particular we have $[\mathbb{P}^{-1}]_{\text{vir}} = 0$ and $[\mathbb{P}^{-2}]_{\text{vir}} = -1$.

Corollary 4.4 *The motivic partition function of the Hilbert scheme of points on a smooth variety X of dimension d equal to 0, 1, 2, or 3 is given by*[6]

$$\sum_{n=0}^{\infty} \left[\text{Hilb}^n(X)\right]_{\text{vir}} T^n = \text{Exp}\big(T[X]_{\text{vir}} \text{Exp}\big(T\big[\mathbb{P}^{d-2}\big]_{\text{vir}}\big)\big),$$

where $T = (-1)^d t$.

Remark 4.5 We do not know of a reasonable general definition for the virtual motive $[\text{Hilb}^n(X)]_{\text{vir}}$ when the dimension of X is greater than 3. However, it is well known that the Hilbert scheme $\text{Hilb}^n(X)$ of $n \leq 3$ points is smooth in all dimensions and so the virtual motive is given by $\mathbb{L}^{-\frac{nd}{2}}[\text{Hilb}^n(X)]$ in these cases (cf. (2.11)). Remarkably, the formula in Corollary 4.4 correctly computes the virtual motive for $n \leq 3$ in all dimensions. This can be verified directly using the motivic class of the punctual Hilbert scheme for $n \leq 3$ [6, §4]:

$$\sum_{n=0}^{3} \left[\text{Hilb}^n(\mathbb{C}^d)_0\right] t^n = 1 + t + \begin{bmatrix} d \\ 1 \end{bmatrix}_{\mathbb{L}} t^2 + \begin{bmatrix} d+1 \\ 2 \end{bmatrix}_{\mathbb{L}} t^3.$$

Remark 4.6 Using the torus action on $\text{Hilb}^n(\mathbb{C}^d)$, one sees that $\chi(\text{Hilb}^n(\mathbb{C}^d))$ counts subschemes given by monomial ideals. Equivalently, $\chi(\text{Hilb}^n(\mathbb{C}^d))$ is equal to the number of dimension d partitions of n. Thus naively, one expects $\chi Z_{\mathbb{C}^d}(T)$ to be the generating function for d dimensional partitions of n, counted with the sign $(-1)^{nd}$. Indeed, this is the case when $d \leq 3$ or when $n \leq 3$. Up to the sign $(-1)^{nd}$, the Euler characteristic specialization of our general formula yields exactly MacMahon's guess for the generating function of dimension d partitions:

$$\chi Z_{\mathbb{C}^d}(T) = \text{Exp}\big((-1)^d t \chi[\mathbb{C}^d]_{\text{vir}} \text{Exp}\big((-1)^d t \chi[\mathbb{P}^{d-2}]_{\text{vir}}\big)\big)$$
$$= \text{Exp}\big(t \, \text{Exp}\big((d-1)t\big)\big)$$

---

[6]We thank Lothar Göttsche, Ezra Getzler and Sven Meinhardt for discussions which led us to this formulation.

$$= \mathrm{Exp}\left(\frac{t}{(1-t)^{d-1}}\right)$$

$$= \prod_{m=1}^{\infty} \left(1 - t^m\right)^{-\binom{m+d-3}{d-2}}.$$

However, it is now known that MacMahon's guess is not correct, although it does appear to be asymptotically correct in dimension four [29].

4.3 Weight polynomial and deformed MacMahon

When the dimension of X is 1 or 2, the weight polynomial specialization of $Z_X(t)$ gives rise to MacDonald's and Göttsche's formulas for the Poincaré polynomials of the Hilbert schemes. When the dimension of X is 3, the weight polynomial specialization leads to the following analogous formula, involving the refined MacMahon functions discussed in Appendix A.

Theorem 4.7 *Let X be a smooth projective threefold and let b_d be the Betti number of X of degree d. Then the generating function of the virtual weight polynomials of the Hilbert schemes of points of X is given by*

$$WZ_X(t) = \prod_{d=0}^{6} M_{\frac{d-3}{2}}\left(-t, -q^{\frac{1}{2}}\right)^{(-1)^d b_d} \in \mathbb{Z}\left[q^{\pm\frac{1}{2}}\right][[t]], \qquad (4.4)$$

where

$$M_\delta\left(t, q^{\frac{1}{2}}\right) = \prod_{m=1}^{\infty} \prod_{k=0}^{m-1} \left(1 - q^{\delta+\frac{1}{2}+k-\frac{m}{2}} t^m\right)^{-1}$$

are the refined MacMahon functions discussed in Appendix A.

Proof Recall that the weight polynomial specialization

$$W : \mathcal{M}_{\mathbb{C}} \to \mathbb{Z}\left[q^{\pm\frac{1}{2}}\right]$$

is obtained from the E polynomial specialization

$$E : \mathcal{M}_{\mathbb{C}} \to \mathbb{Z}\left[x, y, (xy)^{-\frac{1}{2}}\right]$$

by setting $x = y = -q^{\frac{1}{2}}$ and $(xy)^{\frac{1}{2}} = q^{\frac{1}{2}}$. It follows from [18, Proposition 4] that the W-specialization is a ring homomorphism which respects power

structures where the power structure on $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ satisfies

$$\left(1-t^m\right)^{-\sum_i a_i(-q^{\frac{1}{2}})^i} = \prod_i \left(1-(-q^{\frac{1}{2}})^i t^m\right)^{-a_i}.$$

From Theorem 4.3 we deduce that

$$Z_X(t) = Z_{\mathbb{C}^3}(t)^{\mathbb{L}^{\frac{3}{2}}[X]_{\mathrm{vir}}} = Z_{\mathbb{C}^3}(t)^{[X]}.$$

It then follows from Theorem 3.7 that

$$Z_X(t) = \prod_{m=1}^{\infty} \prod_{k=0}^{m-1} \left(1-(-\mathbb{L}^{\frac{1}{2}})^{2k+2-m}(-t)^m\right)^{-[X]}.$$

Applying the homomorphism W to Z_X, using the compatibility of the power structures, we get

$$WZ_X(t) = \prod_{m=1}^{\infty} \prod_{k=0}^{m-1} \left(1-(-q^{\frac{1}{2}})^{2k+2-m}(-t)^m\right)^{-\sum_{d=0}^{6}(-1)^d b_d(-q^{\frac{1}{2}})^d}$$

$$= \prod_{d=0}^{6} \prod_{m=1}^{\infty} \prod_{k=0}^{m-1} \left(1-(-q^{\frac{1}{2}})^{2k+2-m+d}(-t)^m\right)^{-(-1)^d b_d}$$

$$= \prod_{d=0}^{6} M_{\frac{d-3}{2}}\left(-t,-q^{\frac{1}{2}}\right)^{(-1)^d b_d}. \qquad \square$$

Remark 4.8 The Euler characteristic specialization is easily determined from the formula in Theorem 4.7 by setting $-q^{\frac{1}{2}} = 1$, namely

$$\chi Z_X(t) = M(-t)^{\chi(X)}. \qquad (4.5)$$

By Proposition 2.16, this is the partition function of the ordinary degree zero Donaldson–Thomas invariants in the case when X is Calabi–Yau. Formula (4.5) is a result, for any smooth quasi-projective threefold, of Behrend and Fantechi [2].

Note that a variant of this formula, for a smooth projective threefold X, was originally conjectured by Maulik, Nekrasov, Okounkov and Pandharipande [27]. This involves the integral (degree) of the degree zero virtual cycle on the Hilbert scheme, and says that for a projective threefold X,

$$\sum_{n=0}^{\infty} \deg\left[\mathrm{Hilb}^n(X)\right]^{\mathrm{vir}} t^n = M(-t)^{\int_X c_3 - c_1 c_2}, \qquad (4.6)$$

with c_i being the Chern classes of X. This was proved by [23, 24]. The two formulae become identical in the projective Calabi–Yau case, since then the perfect obstruction theory on the Hilbert schemes is symmetric [2], and so by the main result of [1], the degree of the virtual zero-cycle is equal to its virtual Euler characteristic. Note that our work has nothing to say about formula (4.6) in the non-Calabi–Yau case.

4.4 Categorified Donaldson–Thomas invariants

Our definition of the virtual motive of the Hilbert scheme $\mathrm{Hilb}^n(X)$ of a smooth quasi-projective threefold X, obtained by building it up from pieces on strata, is certainly not ideal. Our original aim in this project was in fact to build a categorification of Donaldson–Thomas theory on the Hilbert scheme of a (Calabi–Yau) threefold X, defining an object of some category with a cohomological functor to (multi)graded vector spaces, whose Euler characteristic gives the degree zero Donaldson–Thomas invariant of X. Finding a ring with an Euler characteristic homomorphism is only a further shadow of such a categorification.

One particular candidate where such a categorification could live would be the category $\mathrm{MHM}(\mathrm{Hilb}^n(X))$ of mixed Hodge modules [33, 34] on the Hilbert scheme. This certainly works for affine space $\mathbb{C}^3$, since the global description of the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C}^3)$ as a degeneracy locus gives rise to a mixed Hodge module of vanishing cycles, with all the right properties [12]. However, when trying to globalize this construction, we ran into glueing issues which we couldn't resolve, arising from the fact that the description of $\mathrm{Hilb}^n(\mathbb{C}^3)$ as a degeneracy locus uses a linear Calabi–Yau structure on $\mathbb{C}^3$ and is therefore not completely canonical.

In some particular cases, we were able to construct the mixed Hodge modules categorifying Donaldson–Thomas theory of the Hilbert scheme. Since we currently have no application for categorification as opposed to a refined invariant taking values in the ring of motivic weights, and since our results are partial, we only sketch the constructions.

– *Low number of points.* For $n \leq 3$, the Hilbert scheme $\mathrm{Hilb}^n(X)$ is smooth and there is nothing to do. The next case $n = 4$ is already interesting. It is known that for a threefold X, the space $\mathrm{Hilb}^4(X)$ is irreducible and reduced, singular along a copy of X which is the locus of squares of maximal ideals of points. As proved in [12], for $X = \mathbb{C}^3$ the mixed Hodge module of vanishing cycles of f_4 on $\mathrm{Hilb}^4(\mathbb{C}^3)$ admits a very natural geometric description: it has a three-step non-split filtration with quotients being (shifted copies of) the constant sheaf on the (smooth) singular locus, the intersection cohomology (IC) sheaf of the whole irreducible space $\mathrm{Hilb}^4(\mathbb{C}^3)$, and once more the constant sheaf on the singular locus. It also follows from results of [ibid.] that the relevant extension groups are one-dimensional,

and so this mixed Hodge module is unique. Turning to a general (simply-connected) X, we again have the IC sheaf on the space $\mathrm{Hilb}^4(X)$ and the constant sheaf on its singular locus, and a compatible extension of these mixed Hodge modules exists and is unique. This provides the required categorification. We expect that such an explicit construction is possible for some higher values of n than 4 but certainly not in general.

– *Abelian threefolds.* Let X be an abelian threefold (or some other quotient of $\mathbb{C}^3$ by a group of translations). Then we can cover X by local analytic patches with transition maps which are in the affine linear group of $\mathbb{C}^3$. The local (analytic) vanishing cycle sheaves on the Hilbert schemes of patches can be glued using the affine linear transition maps to a global (analytic) mixed Hodge module on $\mathrm{Hilb}^n(X)$.

– *Local toric threefolds.* Finally, it should be possible to construct the gluing directly for some local toric threefolds. We checked the case of local $\mathbb{P}^1$ explicitly, in which case the mixed Hodge modules on all Hilbert schemes exist. However, we already failed for local $\mathbb{P}^2$.

Compare also the discussion surrounding [20, Question 5.5], and see also the recent work [22].

Appendix A: q-Deformations of the MacMahon function

Let $\mathcal{P}$ denote the set of all finite 3-dimensional partitions. For a partition $\alpha \in \mathcal{P}$, let $w(\alpha)$ denote the number of boxes in α. The combinatorial generating series

$$M(t) = \sum_{\alpha \in \mathcal{P}} t^{w(\alpha)}$$

was determined in closed form by MacMahon [26] to be

$$M(t) = \prod_{m=1}^{\infty} \left(1 - t^m\right)^{-m}.$$

Motivated by work of Okounkov and Reshetikhin [31], in a recent paper [19], Iqbal–Kozçaz–Vafa discussed a family of q-deformations of this formula. Think of a 3-dimensional partition $\alpha \in \mathcal{P}$ as a subset of the positive octant lattice $\mathbb{N}^3$, and break the symmetry by choosing one of the coordinate directions. Define $w_-(\alpha)$, $w_0(\alpha)$ and $w_+(\alpha)$, respectively, as the number of boxes (lattice points) in $\alpha \cap \{x - y < 0\}$, $\alpha \cap \{x - y = 0\}$ and $\alpha \cap \{x - y > 0\}$. For a half-integer $\delta \in \frac{1}{2}\mathbb{Z}$, consider the generating series

$$M_\delta(t_1, t_2) = \sum_{\alpha \in \mathcal{P}} t_1^{w_-(\alpha)+(\frac{1}{2}+\delta)w_0(\alpha)} t_2^{w_+(\alpha)+(\frac{1}{2}-\delta)w_0(\alpha)}.$$

Clearly $M_\delta(t, t) = M(t)$ for all δ.

Theorem A.1 (Okounkov–Reshetikhin [31, Theorem 2]) *The series $M_\delta(t_1, t_2)$ admits the product form*

$$M_\delta(t_1, t_2) = \prod_{i,j=1}^{\infty} \left(1 - t_1^{i-\frac{1}{2}+\delta} t_2^{j-\frac{1}{2}-\delta}\right)^{-1}.$$

In the main body of the paper, we use a different set of variables. Namely, we set

$$t_1 = t q^{\frac{1}{2}}, \qquad t_2 = t q^{-\frac{1}{2}}.$$

Then the product formula becomes

$$M_\delta\left(t, q^{\frac{1}{2}}\right) = \prod_{m=1}^{\infty} \prod_{k=0}^{m-1} \left(1 - t^m q^{k+\frac{1}{2}-\frac{m}{2}+\delta}\right)^{-1}.$$

The specialization to the MacMahon function is $M_\delta(t, q^{\frac{1}{2}} = 1) = M(t)$ for all δ.

Appendix B: The motivic nearby fiber of an equivariant function

In this appendix we prove Proposition 2.12, which asserts that if a regular function $f : X \to \mathbb{C}$ on a smooth variety is equivariant with respect to a torus action satisfying certain assumptions, then Denef-Loeser's motivic nearby fiber $[\psi_f]$ is simply equal to the motivic class of the geometric fiber $[f^{-1}(1)]$. To make this appendix self-contained, we recall the definitions and restate the result below.

Let

$$f : X \to \mathbb{C}$$

be a regular function on a smooth quasi-projective variety X, and let $X_0 = f^{-1}(0)$ be the central fiber. Denef and Loeser define $[\psi_f] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$, the motivic nearby cycle of f using arc spaces and the motivic zeta function [9, 25]. Using motivic integration, they give an explicit formula for $[\psi_f]$ in terms of any embedded resolution which we now recall.

Let $h : Y \to X$ be an embedded resolution of X_0, namely Y is non-singular and

$$\widetilde{f} = h \circ f : Y \to \mathbb{C}$$

has central fiber Y_0 which is a normal crossing divisor with non-singular components $\{E_j : j \in J\}$. For $I \subset J$, let

$$E_I = \bigcap_{i \in I} E_i$$

and let

$$E_I^o = E_I - \bigcup_{j \in I^c} (E_j \cap E_I).$$

By convention, $E_\emptyset = Y$ and $E_\emptyset^o = Y - Y_0$.

Let N_i be the multiplicity of E_i in the divisor $\widetilde{f}^{-1}(0)$. Letting

$$m_I = gcd(N_i)_{i \in I},$$

there is a natural etale cyclic μ_{m_I}-cover

$$\widetilde{E}_I^o \to E_I^o.$$

The formula of Denef and Loeser for the (absolute) motivic nearby cycles of f is given by

$$[\psi_f] = \sum_{I \neq \emptyset} (1 - \mathbb{L})^{|I|-1} \big[\widetilde{E}_I^o, \mu_{m_I} \big] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}. \tag{B.1}$$

Since all the E_I^o appearing in the above sum have natural maps to X_0, the above formula determines the *relative* motivic nearby cycle $[\psi_f]_{X_0} \in \mathcal{M}_{X_0}^{\hat{\mu}}$. The relative motivic vanishing cycle is supported on $Z = \{df = 0\}$, the degeneracy locus of f:

$$[\varphi_f]_Z = [\psi_f]_{X_0} - [X_0]_{X_0} \in \mathcal{M}_Z^{\hat{\mu}} \subset \mathcal{M}_{X_0}^{\hat{\mu}}.$$

Recall that an action of $\mathbb{C}^*$ on a variety V is *circle compact*, if the fixed point set $V^{\mathbb{C}^*}$ is compact and moreover, for all $v \in V$, the limit $\lim_{t \to 0} t \cdot y$ exists.

The following is a restatement of Proposition 2.12 and Proposition 2.13. It is the main result of this appendix.

Theorem B.1 *Let $f : X \to \mathbb{C}$ be a regular morphism on a smooth quasi-projective complex variety. Let $Z = \{df = 0\}$ be the degeneracy locus of f and let $Z_{\mathrm{aff}} \subset X_{\mathrm{aff}}$ be the affinization of Z and X respectively. Assume that there exists an action of a connected complex torus T on X so that f is T-equivariant with respect to a primitive character $\chi : T \to \mathbb{C}^*$, namely $f(t \cdot x) = \chi(t) f(x)$ for all $x \in X$ and $t \in T$. We further assume that there exists a one parameter subgroup $\mathbb{C}^* \subset T$ such that the induced action is circle compact. Then the motivic nearby cycle class $[\psi_f]$ is in $\mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ and is equal to $[X_1] = [f^{-1}(1)]$. Consequently the motivic vanishing cycle class $[\varphi_f]$ is given by*

$$[\varphi_f] = \left[f^{-1}(1) \right] - \left[f^{-1}(0) \right].$$

If we further assume that X_0 is reduced then $[\varphi_f]_{Z_{\mathrm{aff}}}$, the motivic vanishing cycle, considered as a relative class on Z_{aff}, lies in the subring $\mathcal{M}_{Z_{\mathrm{aff}}} \subset \mathcal{M}_{Z_{\mathrm{aff}}}^{\hat{\mu}}$.

By equivariant resolution of singularities [36, Corollary 7.6.3], we may assume that $h : Y \to X$, the embedded resolution of X_0, is T-equivariant. Namely Y is a non-singular T-variety and

$$\widetilde{f} = h \circ f : Y \to \mathbb{C}$$

is T-equivariant with central fiber E which is a normal crossing divisor with non-singular components E_j, $j \in J$. Let $\mathbb{C}^* \subset T$ be the one-parameter subgroup whose action on X is circle compact. Then the action of $\mathbb{C}^*$ on Y is circle compact (since h is proper) and each E_j is invariant (but not necessarily fixed).

We will make use of the Białynicki-Birula decomposition for smooth varieties [3]. This result states that if V is a smooth projective variety with a $\mathbb{C}^*$-action, then there is a locally closed stratification:

$$V = \bigcup_F Z_F$$

where the union is over the components of the fixed point locus and $Z_F \to F$ is a Zariski locally trivial affine bundle. The rank of the affine bundle $Z_F \to F$ is given by

$$n(F) = \mathrm{index}(N_{F/V})$$

where the index of the normal bundle $N_{F/V}$ is the number of positive weights of the fiberwise action of $\mathbb{C}^*$. The morphisms $Z_F \to F$ are defined by $x \mapsto$

$\lim_{t \to 0} t \cdot x$ and consequently, the above stratification also exists for smooth varieties with a circle compact action. As a corollary of the Białynicki-Birula decomposition, we get the following relation in the ring of motivic weights.

Lemma B.2 *Let V be a smooth quasi-projective variety with a circle compact $\mathbb{C}^*$-action. For each component F of the fixed point locus, we define the index of F, denoted by $n(F)$, to be the number of positive weights in the action of $\mathbb{C}^*$ on $N_{F/V}$. Then in $\mathcal{M}_{\mathbb{C}}$ we have*

$$[V] = \sum_F \mathbb{L}^{n(F)}[F],$$

where the sum is over the components of the fixed point locus and $\mathbb{L} = [\mathbb{A}^1_{\mathbb{C}}]$ is the Lefschetz motive.

We call this decomposition a *BB decomposition*.
We begin our proof of Theorem B.1 with a "no monodromy" result.

Lemma B.3 *The following equation holds in $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$:*

$$\left[\widetilde{E}_I^o, \mu_{m_I}\right] = \left[E_I^o\right].$$

Under the further assumption that X_0 is reduced, the above equation holds in $\mathcal{M}_{X_{\mathrm{aff}}}^{\hat{\mu}}$.

An immediate corollary of Lemma B.3 is that $[\psi_f]$ lies in the subring $\mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ and that if X_0 is reduced, then $[\varphi_f]_{Z_{\mathrm{aff}}}$ lies in the subring $\mathcal{M}_{Z_{\mathrm{aff}}} \subset \mathcal{M}_{Z_{\mathrm{aff}}}^{\hat{\mu}}$.

To prove Lemma B.3, we recall the construction of the μ_{m_I}-cover $\widetilde{E}_I^o \to E_I^o$ given in [25, §5]. Let $N = lcm(N_i)$ and let $\widetilde{Y} \to Y$ be the μ_N-cover obtained by base change over the Nth power map $(\cdot)^N : \mathbb{C} \to \mathbb{C}$ followed by normalization. Define $\widetilde{E}_I^o$ to be any connected component of the preimage of E_I^o in $\widetilde{Y}$. The component $\widetilde{E}_I^o$ is stabilized by $\mu_{m_I} \subset \mu_N$ whose action defines the cover $\widetilde{E}_I^o \to E_I^o$.

Observe that the composition

$$E_i \to Y \to X \to X_{\mathrm{aff}}$$

contracts E_i to a point unless E_i is a component of the proper transform of X_0. Thus for these components (and their intersections), the proof given below (stated for absolute classes), applies to relative classes over X_{aff} as

well. Under the assumption that X_0 is reduced, those E_i which are components of the proper transform of X_0 have multiplicity one and so $\widetilde{E}_i = E_i$ and there is nothing to prove.

We define $\widetilde{T}$ by the fibered product

$$
\begin{array}{ccc}
\widetilde{T} & \xrightarrow{\ \widetilde{\chi}\ } & \mathbb{C}^* \\
\downarrow & & \downarrow {\scriptstyle (\cdot)^N} \\
T & \xrightarrow{\ \chi\ } & \mathbb{C}^*
\end{array}
$$

Thus $\widetilde{T}$ is an extension of T by $\mu_N \subset \mathbb{C}^*$ and it has character $\widetilde{\chi}$ satisfying $\widetilde{\chi}^N = \chi$. Moreover, $\widetilde{\chi}$ is the identity on the subgroup $\mu_N \subset \widetilde{T}$. The key fact here is that since χ is primitive, $\widetilde{T}$ is connected.

By construction, $\widetilde{T}$ acts on the base change of Y over the Nth power map and hence it acts on the normalization $\widetilde{Y}$. Thus we have obtained an action of a connected torus $\widetilde{T}$ on $\widetilde{Y}$ covering the T-action on Y. Since T acts on each E_I^o, $\widetilde{T}$ acts on each component of the preimage of E_I^o in $\widetilde{Y}$. Thus we have an action of the connected torus $\widetilde{T}$ on $\widetilde{E}_I^o$ such that the μ_{m_I}-action is induced by the subgroup

$$
\mu_{m_I} \subset \mu_N \subset \widetilde{T}.
$$

Lemma B.3 then follows from the following:

Lemma B.4 *Let W be a smooth quasi-projective variety with the action of a connected torus $\widetilde{T}$. Then for any finite cyclic subgroup $\mu \subset \widetilde{T}$, the equation*

$$
[W, \mu] = [W] = [W/\mu]
$$

holds in $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$.

Proof Let $\mathbb{C}^* \subset \widetilde{T}$ be the 1-parameter subgroup generated by $\mu \subset \widetilde{T}$. The $\mathbb{C}^*$-action on W gives rise to a $\mathbb{C}^*$-equivariant stratification of W into varieties W_i of the form $(V - \{0\}) \times F$ where F is fixed and V is a $\mathbb{C}^*$-representation. This assertion follows from applying the Białynicki-Birula decomposition to $\overline{W}$, any $\mathbb{C}^*$-equivariant smooth compactification of W and stratifying further to trivialize all the bundles and to make the zero sections separate strata. The induced stratification of W then is of the desired form. Thus to prove Lemma B.4, it then suffices to prove it for the case of μ acting on $V - \{0\}$ where V is a μ-representation. By the relation given in (2.4), we have $[V, \mu] = [V]$ and hence $[V - \{0\}, \mu] = [V - \{0\}]$. The equality $[V - \{0\}] = [(V - \{0\})/\mu]$ follows from [25, Lemma 5.1].

Applying Lemma B.4 to (B.1), we see that in order to prove $[\psi_f] = [X_1]$, we must prove

$$[X_1] = \sum_{I \neq \emptyset} (1 - \mathbb{L})^{|I|-1} [E_I^o].$$

As explained in the beginning of Sect. 2.7, there is an isomorphism

$$X_1 \times \mathbb{C}^* \cong X - X_0.$$

Consequently we get $(\mathbb{L} - 1)[X_1] = [X] - [X_0]$ or equivalently

$$[Y] = [Y_0] + (\mathbb{L} - 1)[X_1].$$

Combining this with the previous equation we find that the equation we wish to prove, $[\psi_f] = [X_1]$, is equivalent to

$$[Y] = [Y_0] - \sum_{I \neq \emptyset} (1 - \mathbb{L})^{|I|} [E_I^o],$$

which can also be written (using the conventions about the empty set) as

$$0 = \sum_{I} (1 - \mathbb{L})^{|I|} [E_I^o]. \tag{B.2}$$

By the principle of inclusion/exclusion, we can write

$$[E_I^o] = [E_I] - \sum_{\emptyset \neq K \subset I^c} (-1)^{|K|} [E_{I \cup K}] = \sum_{K \subset I^c} (-1)^K [E_{I \cup K}],$$

thus we have

$$(-1)^{|I|} [E_I^o] = \sum_{A \supset I} (-1)^{|A|} [E_A].$$

Thus the right hand side of (B.2) becomes

$$\sum_{I} (\mathbb{L} - 1)^{|I|} \sum_{A \supset I} (-1)^{|A|} [E_A] = \sum_{A} (-1)^{|A|} [E_A] \sum_{I \subset A} (\mathbb{L} - 1)^{|I|}.$$

Since

$$\sum_{I \subset A} (\mathbb{L} - 1)^{|I|} = \sum_{n=0}^{|A|} \binom{|A|}{n} (\mathbb{L} - 1)^n = \mathbb{L}^{|A|},$$

we can reformulate the equation we need to prove as

$$0 = \sum_{A} (-\mathbb{L})^{|A|} [E_A]. \tag{B.3}$$

Note that each E_A is smooth and $\mathbb{C}^*$ invariant and that the induced $\mathbb{C}^*$-action on E_A is circle compact, so we have a BB decomposition for each.

Let F denote a component of the fixed point set $Y^{\mathbb{C}^*}$ and let $\theta_{F,A}$ denote a component of $F \cap E_A$. Since E_A is smooth and $\mathbb{C}^*$ invariant and the induced $\mathbb{C}^*$-action is circle compact, E_A admits a BB decomposition

$$[E_A] = \sum_F \sum_{\theta_{F,A}} \mathbb{L}^{n(\theta_{F,A})}[\theta_{F,a}]$$

where

$$n(\theta_{F,A}) = \mathrm{index}(N_{\theta_{F,A}/E_A}).$$

Therefore the sum in (B.3) (which we wish to prove is zero) is given by

$$\sum_A (-\mathbb{L})^{|A|}[E_A] = \sum_A \sum_F \sum_{\theta_{F,A}} (-1)^{|A|}\mathbb{L}^{|A|+n(\theta_{F,A})}[\theta_{F,A}].$$

We define a set $I(F)$ by

$$I(F) = \{i : F \subset E_i\}.$$

Then clearly

$$\theta_{F,A} = \theta_{F,A'} \quad \text{if } A \cup I(F) = A' \cup I(F).$$

So writing $A = B \cup C$ where $B \subset I(F)$ and $C \subset I(F)^c$, we can rewrite the above sum as

$$\sum_F \sum_{C \subset I(F)^c} \sum_{\theta_{F,C}} (-\mathbb{L})^{|C|}[\theta_{F,C}] \sum_{B \subset I(F)} (-1)^{|B|}\mathbb{L}^{|B|+n(\theta_{F,B\cup C})}. \tag{B.4}$$

We will show that the inner most sum is always zero which will prove Theorem B.1.

Let $y \in \theta_{F,A}$ where $A = B \cup C$. We write the $\mathbb{C}^*$ representation $T_y Y$ in two ways:

$$TF + N_{F/Y} = TE_A + N_{E_A/Y} = T\theta_{F,A} + N_{\theta_{F,A}/E_A} + \sum_{i \in A} N_{E_i/Y}$$

where restriction to the point y is implicit in the above equation. Counting positive weights on each side, we get

$$n(F) = n(\theta_{F,A}) + \sum_{i \in A} m_i(\theta_{F,A})$$

where

$$m_i(\theta_{F,A}) = \text{index}(N_{E_i/Y}|_{\theta_{F,A}}).$$

Note that $m_i(\theta_{F,A})$ is 1 or 0 depending on if the weight of the $\mathbb{C}^*$-action on $E_i|_y$ is positive or not. Note also that if $i \in I(F)$ then

$$m_i(\theta_{F,A}) = m_{i,F} = \text{index}(N_{E_i/Y}|_F).$$

Moreover, if $i \in I(F)^c$, then

$$N_{E_i/Y}|_y \subset TF|_y$$

and so $m_i(\theta_{F,A}) = 0$.

Thus writing $A = B \cup C$ with $B \subset I(F)$ and $C \subset I(F)^c$, we get

$$n(\theta_{F,A}) = n(F) - \sum_{i \in B} m_{i,F},$$

and so

$$\sum_{B \subset I(F)} (-1)^{|B|} \mathbb{L}^{|B|+n(\theta_{F,A})} = \mathbb{L}^{n(F)} \sum_{B \subset I(F)} (-1)^{|B|} \mathbb{L}^{\sum_{i \in B}(1-m_{i,F})}.$$

For $k = 0, 1$, we define

$$I_k(F) = \big\{ i \in I(F) : m_{i,F} = k \big\}.$$

Then

$$\sum_{B \subset I(F)} (-1)^{|B|} \mathbb{L}^{\sum_{i \in B}(1-m_{i,F})} = \sum_{B_0 \subset I_0(F)} (-\mathbb{L})^{|B_0|} \sum_{B_1 \subset I_1(F)} (-1)^{|B_1|},$$

but

$$\sum_{B_1 \subset I_1(F)} (-1)^{|B_1|} = 0$$

unless $I_1(F) = \emptyset$. Since the above equation implies that the expression in (B.4) is zero, and that in turn verifies (B.2) and (B.3) which are equivalent to Theorem B.1, it only remains for us to prove that $I_1(F) \neq \emptyset$ for all F.

Let $y \in F$. We need to show that for some i, the action of $\mathbb{C}^*$ on $N_{E_i/Y}|_y$ has positive weight.

By the Luna slice theorem, there is an etale local neighborhood of $y \in Y$ which is equivariantly isomorphic to $T_y Y$. Over the point y, we have a decomposition

$$TY = TF + N_{F/E_{I(F)}} + \sum_{i \in I(F)} N_{E_i/Y}.$$

Let $(u_1, \ldots, u_s, v_1, \ldots, v_p, \{w_i\}_{i \in I(F)})$ be linear coordinates on $T_y Y$ compatible with the above splitting. The action of $t \in \mathbb{C}^*$ on $T_y Y$ is given by

$$t \cdot (u, v, w) = \left(u_1, \ldots, u_s, t^{a_1} v_1, \ldots, t^{a_p} v_p, \left\{t^{b_i} w_i\right\}_{i \in I(F)}\right).$$

In these coordinates, the function $\widetilde{f}$ is given by

$$\widetilde{f}(u, v, w) = g(u) \prod_{i \in I(F)} w_i^{N_i}$$

where $g(u)$ is a unit. Since the $\mathbb{C}^*$-action on Y is circle compact, $\widetilde{f}$ is equivariant with respect to an action on $\mathbb{C}$ of positive weight l, that is

$$\widetilde{f}\big(t \cdot (u, v, w)\big) = t^l \widetilde{f}(u, v, w).$$

This implies that

$$l = \sum_{i \in I(F)} b_i N_i.$$

Then since $N_i > 0$ and $l > 0$ we have that $b_i > 0$ for some $i \in I(F)$ and so for this i, we have $m_{i, F} = 1$ which was what we needed to prove. The proof of Theorem B.1 is now complete. $\qquad \square$

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