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BOOK REVIEWS

RATIONAL AND NEARLY RATIONAL VARIETIES
(Cambridge Studies in Advanced Mathematics 92)

By JÁNOS KOLLÁR, KAREN E. SMITH and ALESSIO CORTI:
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Consider a smooth affine quadric curve over the complex numbers: a plane curve in the complex affine space $\mathbb{A}^2_{\mathbb{C}}$ given as the locus of points (x, y) with $Q(x, y) = 0$, where Q is a non-degenerate two-variable polynomial of total degree two. As was already well known to the classical geometers, this curve admits an algebraic parametrization: there are algebraic expressions $x(t)$, $y(t)$ of a complex variable t such that the points of the curve are given by $(x(t), y(t))$ for $t \in \mathbb{C}$. To see a specific example, consider

$$Q(x, y) = x^2 + y^2 - 1;$$

a parametrization of the corresponding curve is given by

$$x(t) = \frac{2t}{t^2 + 1}, \qquad y(t) = \frac{t^2 - 1}{t^2 + 1}.$$

In contrast, consider the cubic complex curve given as the vanishing locus of the cubic polynomial

$$R(x, y) = y^2 - x^3 - x.$$

It is a classical result that this curve also admits a parametrization by functions of a complex variable z. This time the expressions are

$$x(z) = \mathcal{P}(z), \qquad y(z) = \mathcal{P}'(z)$$

where $\mathcal{P}(z)$ is the Weierstrass $\mathcal{P}$-function associated with a specific lattice in $\mathbb{C}$. This parametrization, however, is not given by algebraic functions; indeed, it is not hard to prove that a smooth cubic plane curve cannot be algebraically parametrized. In short, a smooth quadric curve over the complex numbers is rational, whereas a smooth cubic curve is not.

The book under review studies the rationality, 'near-rationality' or otherwise of higher-dimensional varieties, defined over arbitrary fields. Indeed, much of the effort is expended on formulating results and proofs in birational algebraic geometry which work over fields which are possibly not algebraically closed, and possibly of positive characteristic. To quote the basic definition, an n-dimensional algebraic variety X defined over a field k is *rational*, if it allows a birational map

$$\phi\colon \mathbb{P}^n \dashrightarrow X,$$

defined over k. In other words, X can be algebraically parametrized by polynomials of n variables, with coefficients in k, in an essentially one-to-one way. 'Nearly rational' is not a precise technical term; it refers to various classes of varieties which share many characteristics with rational varieties. Indeed, an important theme of the book is the difficulty involved in showing that a particular nearly rational variety is, in fact, not rational.

The introductory chapter gives a summary of the questions, methods and constructions studied in the subject: quadrics, cubics, finite fields, singular varieties, linear series, rational points and differential forms all make an appearance. The authors do not shy away from explicit examples; indeed, a careful analysis of a particularly interesting real cubic surface, due to Swinnerton-Dyer, is given, which has two connected components, one of which has a dense set of points defined over the rational field $\mathbb{Q}$, while the other has no such points at all. The next chapter treats cubic surfaces in more detail. A basic theorem, due to Segre with an important improvement due to Manin, is proved: *a cubic surface $S \subset \mathbb{P}^3$, on which every curve is linearly equivalent to a hyperplane section, is not rational; furthermore, two such surfaces, defined over perfect fields, are birational to each other if and only if they are projectively equivalent in $\mathbb{P}^3$.* The proofs involve a clever argument involving the linear series defining a possible rational map on S. Similar, though more involved, arguments are used to prove one of the oldest and most discussed results in birational geometry: the theorem of Noether and Castelnuovo, stating that any birational self-map ϕ of the projective plane $\mathbb{P}^2$ over an algebraically closed field is a composite of projective linear transformations and standard quadratic transformations

$$(x_0 : x_1 : x_2) \mapsto (x_0^{-1} : x_1^{-1} : x_2^{-1}).$$

Pitfalls in early 'proofs' of this result illustrate the subtlety of the subject: although it is relatively easy to give a 'proof in outline' arguing on multiplicities of base points of the linear series defining ϕ, a full proof necessarily involves some rather intricate descending induction, and careful analysis of sets of infinitely near base points. The proof given in the book is phrased in the Sarkisov language: a birational self-map of $\mathbb{P}^2$ is broken up into a series of simpler 'links', and the descending induction involves two integer invariants associated to the singularities of a birational map. Another indication of the subtlety of the situation is, of course, the fact that a corresponding result for $\mathbb{P}^3$, describing a set of generators for the birational automorphism group, appears to be way out of reach at present.

Chapter 3 treats another classical subject, that of rational surfaces. A particular highlight of the chapter is a detailed discussion of Del Pezzo surfaces, defined by the condition that the anticanonical system is ample: some multiple has enough sections to embed the surface in some projective space. Once again, the statements are formulated carefully for non-closed fields.

Chapters 4 and 5 form the technical core of the book, as far as the question of non-rationality is concerned. In Chapter 4, the emphasis is on characteristic p methods, and non-rationality using properties of differential forms. The introductory Chapter 1 had already explained that on rational varieties, there are no non-zero global differential m-forms and k-canonical forms (sections of the kth power of the canonical line bundle). However, several classes of nearly rational varieties satisfy this condition, so more refined methods are needed. The heart of the idea involves p-fold covers Z of projective space $\mathbb{P}^n$ in characteristic p, and a very special subsheaf Q of the sheaf of differential 1-forms Ω_Z. It can be proved that under suitable numerical assumptions, some exterior power of Q is a big line bundle (some power of it gives a birational map on Z). This would already imply that Z cannot be rational, and in fact cannot even be uni-ruled (a weaker notion of 'algebraically parametrized'). An additional complication arises from the fact that the variety Z used in the construction is singular; however, it can be desingularized using point blowups only, and hence the situation is tractable. Once the non-uniruledness of Z is established, it is relatively easy to produce examples of non-rational, nearly rational varieties in characteristic zero by a specialization argument.

Chapter 5 treats a completely different method, applicable in different situations: arguments based around the Noether–Fano inequalities, giving detailed estimates on multiplicities of linear series near their singularities, generalizing the methods already used for surfaces in Chapter 2. The basic idea is that certain birational maps arise from linear systems with a *maximal center*, a subvariety of its base locus along which it is 'very singular'. The computations need care, but the results are very strong; in particular, the theorem of Fano, Manin and Iskovskih, non-rationality of smooth quartic threefolds in $\mathbb{P}^4$, as well as generalizations for weighted projective hypersurfaces, are proved. A technical statement is relegated to Chapter 6, which discusses singularities of pairs (X, D) consisting of a normal variety X and a $\mathbb{Q}$-linear combination D of linear systems on X; this is hardcore birational geometry. The important category of canonical and log-canonical pairs is introduced, and inversion of adjunction, restriction of a pair to a hypersurface, is discussed. All this leads up to a proof of a statement on maximal point centers on smooth threefolds, needed to complete the proof of the non-rationality results of Chapter 5.

As should be clear from this summary, the book begins rather mildly, but takes the adventurous reader far into recently charted and eventually uncharted territory. Several aspects of modern algebraic geometry make an appearance: the minimal model program, the Sarkisov program, characteristic p and graded rings methods, to name but a few. The proofs are detailed, but require active participation from the reader; indeed, they often split up into a series of exercises, many of which have worked solutions in the closing Chapter 7. This book would make an ideal graduate course, which could be taught after a first course on algebraic geometry. However, the professional birational geometer will also learn from the sympathetic treatment given to non-closed and positive characteristic fields, the careful recollection and proofs of classical results, and the user-friendly discussion of non-rationality theorems recently obtained by the authors and others.

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HARMONIC MAPPINGS IN THE PLANE
(Cambridge Tracts in Mathematics 156)

By PETER DUREN: 212 pp., £40.00 (US$60.00), ISBN 0-521-64121-7
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A complex-valued function $f(z) = u(z) + iv(z)$ of a complex variable is called a harmonic function if f is a solution of the Laplace equation componentwise: that is, if $\triangle u = 0 = \triangle v$. Thus a harmonic function enjoys the same properties as a plane analytic function, except that there is no coupling provided by the Cauchy–Riemann equations. Harmonic functions include analytic functions and, especially, conformal mappings. This fact very much dominates the material in the book.

The origin of harmonic mappings has two main sources. In a representation of a minimal surface by the isothermal coordinates, the three coordinate functions are harmonic. This development had a boost in the early 20th century; Tibor Rado, Lipman Bers, Erhard Heinz and Johannes Nietsche made their marks in this research. The other source is relatively new and concerns two-dimensional harmonic mappings, which have many properties not shared by higher-dimensional harmonic mappings. The most influential ideas for current research came from James Clunie and Terry Sheil-Small [1] in 1984. They were the first to employ complex analytic methods to study univalent harmonic mappings. The first eight chapters of the book are mainly concerned with the latter topic, and the last two chapters with minimal surfaces.

Elementary features of the interplay between analytic, conformal, quasiconformal and harmonic functions are explained in the first, preliminary chapter. Complex notation is used