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ORBIFOLD RIEMANN–ROCH FOR THREEFOLDS WITH AN APPLICATION TO CALABI–YAU GEOMETRY

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Abstract

We prove an orbifold Riemann–Roch formula for a polarized complex 3–fold (X, D). As an application, we construct new families of projective Calabi–Yau threefolds.

Introduction

The aim of this paper is to state and prove a Riemann–Roch (RR) formula for a pair (X, D) of a normal projective threefold X with quotient singularities and a $\mathbb{Q}$-Cartier Weil divisor D. Under certain conditions, we prove the existence of a formula

$$\chi(X, \mathcal{O}_X(D)) = \text{ RR–type expression in } D + \sum_P c_P(D) + \sum_C s_C(D),$$

where $c_P(D)$ is the contribution from a singular point P of (X, D) and $s_C(D)$ is the contribution from a curve C of singularities. These contributions depend on the type of the quotient singularities and on the embedding of $\mathrm{Sing}X$ in X.

Explicit formulae of this type for surfaces and threefolds with isolated canonical singularities were first studied by Fletcher [12] and Reid [21]. Earlier Kawasaki [15] had proved a general Riemann–Roch formula for orbifolds. His formula involves a sum over loci in X with constant inertia group and specifies to the above form in our case. Our proof relies on equivariant Riemann–Roch and a computation of intersection numbers on a resolution and is independent of Kawasaki's; it gives the contributions in an explicit form (which could also be deduced from Kawasaki's result with extra work). Kawasaki's analytic

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formula was extended by Toen [23] to the context of Deligne–Mumford stacks using algebraic methods.

Under further conditions on (X, D), vanishing implies that $\chi(X, \mathcal{O}_X(nD))$ is simply the dimension of $H^0(X, \mathcal{O}_X(nD))$ for $n > 0$. The finite-dimensional vector spaces $H^0(X, \mathcal{O}_X(nD))$ fit together into a graded ring

$$R(X, D) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD)).$$

If D is assumed ample as well as being $\mathbb{Q}$-Cartier, this ring is finitely generated. A surjection

$$k[x_0, \ldots, x_n] \twoheadrightarrow R(X, D)$$

from a graded ring $k[x_0, \ldots, x_n]$ generated by variables x_i of weights a_i corresponds to an embedding

$$i\colon X \cong \mathrm{Proj} R(X, D) \hookrightarrow \mathbb{P}(a_0, \ldots, a_n)$$

of X into a weighted projective space, with $\mathcal{O}_X(D)$ isomorphic to $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}}(1)$.

Our aim is to construct quasi-smooth and well-formed threefolds in weighted projective space. This implies in particular that the only singularities of X are quotient singularities induced by the weights of the weighted projective space. Moreover, since we are interested in constructing threefolds with at worst canonical singularities, we restrict our study to curves of singularities which are generically of compound Du Val (cDV) type [19, Definition 2.1].

The main result of this paper is Theorem 2.1, which presents an explicit RR formula of the above shape. As an application, we find new projective families of Calabi–Yau threefolds, via a study of their Hilbert series, which is brought into a compact form in Corollary 3.3. The detailed and exhaustive analysis of families arising in this way, as well as applications to other families of varieties such as Fanos and regular varieties of general type, will be presented elsewhere.

1. Definitions and notation

We work over an algebraically closed field k of characteristic zero. A $\mathbb{Q}$–*divisor* on a normal variety X is a formal linear combination of prime divisors with rational coefficients. A $\mathbb{Q}$–divisor D is $\mathbb{Q}$–*Cartier* if mD is Cartier for some positive integer $m \neq 0$. In this case, if $C \subset X$ is a complete curve, the

degree of D on C is defined by

$$\deg D|_C = \frac{1}{m} \cdot \deg_C \mathcal{O}_X(mD).$$

If X is projective, we also define the intersection $D \cdot c_2(X)$ as $\frac{1}{m} f^*(mD) \cdot c_2(Y)$ computed on a resolution $f\colon Y \to X$, minimal over the cDV locus. A *Calabi–Yau threefold* is a normal projective threefold X with canonical Gorenstein singularities, satisfying $K_X \sim \mathcal{O}_X$ (linear equivalence) and $H^1(X, \mathcal{O}_X) = 0$.

A *cyclic quotient singularity of type* $\frac{1}{r}(b_1, \dots, b_n)$ is the quotient $\pi\colon \mathbb{A}^n \to \mathbb{A}^n/\mu_r$, where μ_r acts on $\mathbb{A}^n$ by

$$\mu_r \ni \varepsilon\colon\ (x_1, \dots, x_n) \mapsto (\varepsilon^{b_1} x_1, \dots, \varepsilon^{b_n} x_n).$$

We always assume that no factor of r divides all the b_i, which is equivalent to the μ_r-action being effective. The sheaf $\pi_* \mathcal{O}_{\mathbb{A}^n}$ decomposes into eigensheaves

$$\mathcal{L}_i = \{f \mid \varepsilon(f) = \varepsilon^i \cdot f \text{ for all } \varepsilon \in \mu_r\},$$

for $i = 0, \dots, r-1$. A singularity $Q \in X$ polarised by a Weil divisor D is a *cyclic quotient singularity of type* ${}_i\big(\frac{1}{r}(b_1, \dots, b_n)\big)$ if $Q \in X$ is locally isomorphic to a point of type $\frac{1}{r}(b_1, \dots, b_n)$ and $\mathcal{O}_X(D) \cong \mathcal{L}_i$ near the singular point.

Let (X, D) be a threefold X containing a curve C of singularities, equipped with a $\mathbb{Q}$-Cartier divisor D which is Cartier away from C. Take a generic surface S which intersects C transversely in a finite number of points. Assume that every point in the intersection is a singular point of type ${}_k\big(\frac{1}{r}(1, -1)\big)$ on the polarised surface $(S, D|_S)$. Then $C \in X$ will be called a *curve of singularities of transverse type* ${}_k\big(\frac{1}{r}(1, -1)\big)$, in short a ${}_k\big(\frac{1}{r}(1, -1)\big)$ *curve*, or often simply an A_{r-1} *curve*.

Note that every A_{r-1} curve can contain a finite number of points of different type, which will be called *dissident points*. A curve with dissident points is a *dissident curve*. If an A_{r-1} curve C contains the dissident points $\{P_\lambda : \lambda \in \Lambda\}$ of types $\frac{1}{r\tau_\lambda}(a_{1\lambda}, a_{2\lambda}, a_{3\lambda})$, then we define the *index of* C to be

$$\tau_C = \operatorname{lcm}_{\lambda \in \Lambda}\{\tau_\lambda\}.$$

For an A_{r-1} curve C with no dissident points, $\tau_C = 1$.

Example 1.1. The degree 13 hypersurface $X_{13} \subset \mathbb{P}(1, 1, 2, 3, 6)$ in the weighted projective space with variables x_1, x_2, y, z, t of the given degrees is a Calabi–Yau threefold with two curves of singularities intersecting in a dissident point. The first $C_1 = \{x_1 = x_2 = y = 0\}$ is of type $\frac{1}{3}(1, 2)$ and index 2 because of the dissident point $(0, 0, 0, 0, 1)$ of type $\frac{1}{6}(1, 2, 3)$; likewise, $C_2 = \{x_1 = x_2 = z = 0\}$ is of type $\frac{1}{2}(1, 1)$ and index 3.

For details on subvarieties in weighted projective space, consult [13]. In particular, recall that a variety $X \subset \mathbb{P}^n(a_0, \ldots, a_n) = \mathbb{P}$ is *quasi-smooth* if the affine cone $C_X \subset \mathbb{A}^{n+1}$ over X is smooth outside its vertex. In this case X only has quotient singularities induced by the singularities of $\mathbb{P}$. The weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$ is *well formed* if

$$(a_0, \ldots, \widehat{a_i}, \ldots, a_n) = 1 \quad \text{for each } i.$$

Moreover, X of codimension c in $\mathbb{P}$ is *well formed* if it does not contain any $(c+1)$-codimensional singular stratum of $\mathbb{P}$. Finally a pair (X, D) of a variety X and a $\mathbb{Q}$-divisor D is quasi-smooth, respectively well formed, if there is an ample Cartier divisor H on X, so that under the embedding $X \hookrightarrow \mathbb{P}$ into a weighted projective space induced by the ample $\mathbb{Q}$-Cartier divisor $D + H$, the pair $X \subset \mathbb{P}$ is quasi-smooth, respectively well formed.

2. The Riemann–Roch formula

2.1. The statement. Here is the main theorem of this paper:

Theorem 2.1. *Let (X, D) be a pair consisting of a normal projective threefold and a $\mathbb{Q}$-Cartier divisor, which is quasi-smooth and well formed. Assume further that the singularities of (X, D) consist of the following loci:*

- *points $P \in X$ of type ${}_n\big(\frac{1}{s}(a_1, a_2, a_3)\big)$ (dissident and isolated), and*
- *curves $C \subset X$ of generic type ${}_k\big(\frac{1}{r}(1, -1)\big)$ with index τ_C.*

Then for all positive integers m,

$$\chi(X, \mathcal{O}_X(mD)) = \chi(\mathcal{O}_X) + \tfrac{1}{12} mD(mD - K_X)(2mD - K_X) + m\frac{D \cdot c_2(X)}{12} + \sum_P c_P(mD) + \sum_C s_C(mD),$$

where

$$c_P(mD) = \frac{1}{s} \sum_{\substack{\varepsilon \in \mu_s \\ \varepsilon^{a_i} \neq 1 \, \forall i = 1,2,3}} \frac{\varepsilon^{-nm} - 1}{(1 - \varepsilon^{-a_1})(1 - \varepsilon^{-a_2})(1 - \varepsilon^{-a_3})}$$

and

$$s_C(mD) = -m\frac{\overline{mk}(r - \overline{mk})}{2r} \deg D|_C + \frac{\overline{mk}(r - \overline{mk})}{4r} \deg K_X|_C + \frac{\overline{mk}(r - \overline{mk})(r - 2 \cdot \overline{mk})}{12 r^2 \tau_C} N_C,$$

where the integer N_C is an invariant of X in a neighbourhood of C, and $\overline{}$ denotes the smallest residue mod r.

Remark 2.2. Note that a point $P \in X$ of type ${}_n\big(\frac{1}{s}(a_1, a_2, a_3)\big)$ is either

(1) an isolated point singularity if $\mathrm{hcf}(a_i, s) = 1$ for all $i = 1, 2, 3$, or
(2) a dissident point on some curve if $\mathrm{hcf}(a_i, s) = \alpha_i \neq 1$ for some i.

In the first case, $c_P(mD)$ equals the so-called basket contribution to Riemann–Roch [21]. In the second case, as the singularities along curves are of $A_{\alpha_i - 1}$ transverse type, we must have $a_j + a_k = 0 \bmod \alpha_i$ for different indexes i, j, k. Since X is well formed, a_i, a_j, s have no common divisor.

2.2. The outline of the proof. Let (X, D) be a pair satisfying the conditions of Theorem 2.1. Choose a projective resolution $f\colon Y \to X$ which is crepant at the generic point of each one-dimensional component of the singular locus of (X, D). The sheaf $\mathcal{O}_X(D)$ is a rank-1 reflexive sheaf on X; define $\mathcal{L} = f^*(\mathcal{O}_X(D))/(\text{torsion})$, a rank-one torsion-free sheaf on Y. Also, let H be an ample Cartier divisor on X.

For some integers n, N, there exists a surjection $\mathcal{O}_X^N(-nH) \twoheadrightarrow \mathcal{O}_X(D)$ on X which pulls back to a surjection $\mathcal{O}_Y^N \otimes f^*\mathcal{O}_X(-nH) \twoheadrightarrow \mathcal{L}$ on Y and hence gives an exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_Y^N \otimes f^*\mathcal{O}_X(-nH) \to \mathcal{L} \to 0.$$

Under f_*, this becomes the long exact sequence

$$\begin{array}{ccccccc} 0 \to & f_*\mathcal{K} & \to & \mathcal{O}_X^N(-nH) & \to & f_*\mathcal{L} & \to \\ & R^1 f_*\mathcal{K} & \to & 0 & \to & R^1 f_*\mathcal{L} & \to \\ & R^2 f_*\mathcal{K} & \to & 0 & \to & R^2 f_*\mathcal{L} & \to 0, \end{array}$$

where we used $f_*\mathcal{O}_Y \cong \mathcal{O}_X$, which holds since X is normal, and the projection formula together with $R^i f_*\mathcal{O}_Y = 0$ for $i > 0$, which holds as quotient singularities in characteristic zero are rational. It is easy to see that $f_*\mathcal{L} \cong \mathcal{O}_X(D)$; from the exact sequence we also have $R^2 f_*\mathcal{L} = 0$ and $R^1 f_*\mathcal{L} \cong R^2 f_*\mathcal{K}$, the latter necessarily supported on the isolated and dissident singularities of (X, D).

The reflexivization of the rank-one torsion-free sheaf $\mathcal{L}$ on the smooth variety Y is a line bundle $\mathcal{O}_Y(D_Y)$. As $\mathcal{O}_X(D)$ is saturated, we also have $f_*\mathcal{O}_Y(D_Y) \cong \mathcal{O}_X(D)$. Thus the exact sequence

$$0 \to \mathcal{L} \to \mathcal{O}_Y(D_Y) \to \mathcal{Q} \to 0$$

gives rise to an injection

$$f_*\mathcal{Q} \hookrightarrow R^1 f_*\mathcal{L} \cong R^2 f_*\mathcal{K},$$

proving that f_*Q is supported on the isolated and dissident singular points of (X, D). As $\mathcal{Q}$ is supported in codimension two on Y, this implies that all its higher push-forwards are also supported on these singular points.

Finally, by the Leray spectral sequence

$$\chi(Y, \mathcal{L}) = \sum_i (-1)^i \chi(X, R^i f_* \mathcal{L}),$$

which can be rewritten, using the above exact sequences and isomorphisms, as

$$\begin{aligned} \chi(X, \mathcal{O}_X(D)) &= \chi(Y, \mathcal{O}_Y(D_Y)) + \chi(X, R^1 f_* \mathcal{L}) \\ &\quad - \sum_i (-1)^i \chi(X, R^i f_* \mathcal{Q}) \\ &= \chi(Y, \mathcal{O}_Y(D_Y)) + \mathcal{P}_1, \end{aligned} \tag{2.1}$$

where, by our earlier remarks, $\mathcal{P}_1$ is a contribution from sheaves supported entirely on the isolated and dissident singular points of (X, D). Our arguments in fact imply that this contribution is local in a stronger sense: it only depends on the analytic type of the isolated and dissident quotient singularities of (X, D). This holds since the constructions of $\mathcal{L}$ and $\mathcal{Q}$ are universal, and an analytic isomorphism preserving the type of the quotient singularities necessarily gives an analytic isomorphism between these sheaves, and thus an equality of Euler characteristics.

In the next step we will express D_Y and K_Y in terms of D, K_X and the exceptional divisors of the resolution. For a singular point $P \in X$, let $\{{}^P G_j\}$ be the exceptional surfaces mapping to P under f; similarly, for a curve $C \subset X$ of singularities, let $\{{}^C E_i\}$ be the exceptional surfaces mapping surjectively to C. Note that every dissident or isolated singular point P is locally analytically isomorphic to $\mathbb{A}^3/\mu_s$, and the configuration of $\{{}^P G_j\}$ depends only on the analytic singularity type of P.

As f is crepant at the generic point of each curve C, we have

$$K_Y = f^* K_X + N, \text{ where } N = \sum_P \sum_j {}^P\gamma_j \, {}^P G_j;$$

also,

$$f^* D = D_Y + \sum_C R_C + M, \text{ where } R_C = \sum_i {}^C\alpha_i \, {}^C E_i \text{ and } M = \sum_P \sum_j {}^P\beta_j \, {}^P G_j,$$

with ${}^C\gamma_j, {}^C\alpha_i, {}^P\beta_j \in \mathbb{Q}$. Here f^*D is by definition $\frac{1}{m} f^*(mD)$ for an integer m which makes mD Cartier.

Lemma 2.3. *If* $\overline{}$ *denotes the smallest residue mod* r*, then*

$$R_C = \frac{k}{r} \, {}^C E_1 + \frac{\overline{2k}}{r} \, {}^C E_2 + \ldots + \frac{\overline{(r-1)k}}{r} \, {}^C E_{r-1}.$$

Proof. Choose a general transverse hypersurface S and a point $Q \in S \cap C$. By assumption, on the polarised $(S, D|_S)$ the type of Q is ${}_k\big(\frac{1}{r}(1,-1)\big)$. Such a point has a unique crepant resolution $f|_{\tilde{S}} \colon \tilde{S} \to S$, and we necessarily have

$$(f|_{\tilde{S}})^* D|_S = D_Y|_{\tilde{S}} + \frac{k}{r}F_1 + \frac{\overline{2k}}{r}F_2 + \ldots + \frac{\overline{(r-1)k}}{r}F_{r-1},$$

where the F_i are exceptional lines in $\tilde{S}$ forming an A_{r-1} configuration.

Each F_i is a fibre in one of the surfaces ${}^C E_j$. Since X is assumed quasi-smooth and well formed, there is no ramification in codimension 1; thus k and r are coprime. Therefore the integers $k, \overline{2k}, \ldots, \overline{(r-1)k}$ are all different, which implies that $F_1, \ldots, F_{r-1}$ must be fibres in different surfaces ${}^C E_1, \ldots, {}^C E_{r-1}$. Indeed, if some F_i would be fibres in the same surface, the relevant coefficients should be equal, arising from an irreducible divisor on the threefold. The statement follows. □

In particular, there are $(r-1)$ irreducible exceptionals ${}^C E_1, \ldots, {}^C E_{r-1}$ over every one-dimensional component C, each isomorphic to a blowup in a few points (over the dissident points) of a smooth surface geometrically ruled over C. Moreover, ${}^C E_i$ and ${}^C E_j$ intersect along a curve isomorphic to C if $j = i+1$ and are disjoint otherwise. The situation is shown in Figure 1. For two different curves C_1 and C_2, the surfaces ${}^{C_1}E_i$ and ${}^{C_2}E_j$ intersect only over the dissident points.

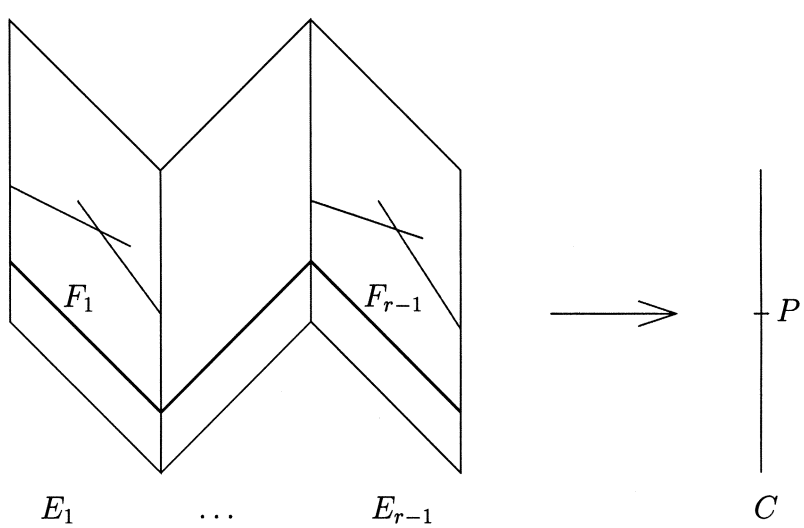

FIGURE 1. Resolution of an A_{r-1} curve C with a dissident point P

We proceed to show that many intersection numbers between these divisors actually vanish. Choose an ample Cartier divisor H on X such that $D+H$ is also ample. Take t large and divisible, so that the linear systems $|\,t\,(D+H)|$ and $|tH|$ contain nonsingular divisors S_1 and S_2 respectively. Then $f^*S_1 \cdot {}^C E_i$ is a finite union of fibres in ${}^C E_i$, since $f^*S_1|_{{}^C E_i}$ is a pull-back of $t(D+H)|_C$; the same holds for $f^*S_2 \cdot {}^C E_i$. Then using also the projection formula,

$$\begin{aligned} f^*(tD) \cdot f^*(tD) \cdot {}^C E_i &= (f^*S_1 - f^*S_2) \cdot (f^*S_1 - f^*S_2) \cdot {}^C E_i \\ &= f^*S_1 \cdot (\text{fibres}) - f^*S_2 \cdot (\text{fibres}) \\ &= S_1 \cdot f_*(\text{fibres}) - S_2 \cdot f_*(\text{fibres}) \\ &= 0. \end{aligned}$$

Thus $f^*D \cdot f^*D \cdot R_C = 0$ and similarly $f^*D \cdot f^*K_X \cdot R_C = 0$. The projection formula also gives

$$(f^*D)^3 = D^3,\ (f^*D)^2 \cdot f^*K_X = D^2 K_X,\ f^*D \cdot (f^*K_X)^2 = DK_X^2.$$

The intersection numbers involving M or N depend only on the isolated or dissident singular points of X. Moreover, for different curves of singularities, the intersections

$$f^*D \cdot R_C R_{\bar{C}},\ f^*K_X \cdot R_C R_{\bar{C}} \ \text{ and } \ (R_C)^2 R_{\bar{C}}$$

can only be nonzero because of intersections over dissident points.

Finally using (2.1), Riemann–Roch for the smooth threefold Y, and the birational invariance of $\chi(\mathcal{O}_X)$, we obtain

$$\begin{aligned} \chi(X, \mathcal{O}_X(D)) = \chi(\mathcal{O}_X) &+ \frac{1}{12} D(D-K_X)(2D-K_X) + \frac{1}{12} f^*D \cdot c_2(Y) \\ &+ \frac{1}{2} f^*D \cdot \sum_C R_C^2 - \frac{1}{4} f^*K_X \cdot \sum_C R_C^2 \\ &- \frac{1}{6} \sum_C R_C^3 - \frac{1}{12} \sum_C R_C \cdot c_2(Y) + \mathcal{P}_2. \end{aligned} \tag{2.2}$$

Here $\mathcal{P}_2$ denotes a quantity that depends, in addition to the earlier quantity $\mathcal{P}_1$, on intersection numbers of divisors over the isolated and dissident singularities of (X, D). Since the latter are purely analytical, $\mathcal{P}_2$ also depends only on the analytic type of these singularities.

2.3. The contribution from a curve of singularities. In this section we focus on the part of the contribution that depends only on a small analytic neighbourhood of the curves of singularities. During the argument we will often meet divisors and intersection numbers which depend on the configuration of exceptional divisors over the dissident points and thus depend only on the analytic type of these points. We will denote all such divisors

by $\mathcal{D}$ and numerical contributions by $\mathcal{P}$ and will not worry about the exact expressions.

The next two lemmas will analyze the intersection numbers in (2.2).

Lemma 2.4. *For a* ${}_k\left(\frac{1}{r}(1,-1)\right)$ *curve* C *of singularities of* (X,D),

$$\frac{1}{2}f^*D\cdot R_C^2 = -\deg D|_C\cdot\frac{\overline{k}(r-\overline{k})}{2r}$$

and

$$\frac{1}{4}f^*K_X\cdot R_C^2 = -\deg K_X|_C\cdot\frac{\overline{k}(r-\overline{k})}{4r}.$$

Proof. If $S\subset X$ is a surface which intersects C transversely in all points, then the points in $C\cap S$ are all of type ${}_k\left(\frac{1}{r}(1,-1)\right)$ on $(S,D|_S)$. Riemann–Roch for a resolution $f|_{\tilde{S}}\colon\tilde{S}\to S$ of these points gives

$$\chi(S,\mathcal{O}_S(D)) = \chi(\mathcal{O}_S)+\frac{1}{2}D|_S(D|_S-K_S)+\frac{1}{2}B_{\overline{k}}^2,$$

as in the proof of Theorem 9.1 in [21]. Here $B_{\overline{k}} = R_C|_{\tilde{S}}$ is the $\mathbb{Q}$-divisor on the surface $\tilde{S}$ contributing the so-called basket contribution

$$\frac{1}{2}B_{\overline{k}}^2 = \#(C\cap S)\cdot\left(-\frac{\overline{k}(r-\overline{k})}{2r}\right).$$

As before, for some integer t we can write tD as a difference of two smooth surfaces S_1,S_2 both intersecting C transverselly; thus

$$\frac{1}{2}f^*D\cdot R_C^2 = \frac{1}{2t}(f^*S_2-f^*S_1)\cdot R_C^2 = -\deg D|_C\cdot\frac{\overline{k}(r-\overline{k})}{2r}.$$

A similar argument also shows that

$$\frac{1}{4}f^*K_X\cdot R_C^2 = -\deg K_X|_C\cdot\frac{\overline{k}(r-\overline{k})}{4r},$$

which ends the proof of Lemma 2.4. □

Lemma 2.5. *Let* C *be a* ${}_k\left(\frac{1}{r}(1,-1)\right)$ *curve in* (X,D). *Then*

$$R_C^3+\tfrac{1}{2}R_C\cdot c_2(Y) =$$

$$-\tfrac{1}{r^2}\overline{k}(r-\overline{k})(r-2\overline{k})\left[(r-2)\left(1-g+\tfrac{1}{2}K_Y\,{}^CE_1\,{}^CE_2\right)+{}^CE_{r-2}^2\,{}^CE_{r-1}\right]+\mathcal{P},$$

where g *is the genus of* C.

Proof. From the definition of R_C,

$$R_C^3 = \sum_{i=1}^{r-1}\left(\frac{\overline{ik}}{r}\right)^3E_i^3+3\sum_{i=1}^{r-2}\left(\frac{\overline{ik}}{r}\right)^2\frac{\overline{(i+1)k}}{r}E_i^2E_{i+1}+\frac{\overline{ik}}{r}\left(\frac{\overline{(i+1)k}}{r}\right)^2E_iE_{i+1}^2$$

and

$$R_C\cdot c_2(Y) = \left(\frac{\overline{k}}{r}E_1+\frac{\overline{2k}}{r}E_2+\ldots+\frac{\overline{(r-1)k}}{r}E_{r-1}\right)\cdot c_2(Y).$$

We will simplify these expressions using the properties of ruled surfaces. Recall that each E_i is isomorphic to a smooth ruled surface $\hat{E}_i$ blown up in a number of points. The blowups $E_i \xrightarrow{\pi_i} \hat{E}_i$ happen only over the dissident points. In other words, the resolution $f\colon Y \to X$ contracts $\mathrm{Exc}(\pi_i) \subset E_i$ to the dissident points on C.

Denote $\gamma_i = E_i \cap E_{i+1}$, which is isomorphic to C of genus g. Note that $\pi_i(\gamma_i) = \pi_{i+1}(\gamma_i)$ is a section in both ruled surfaces $\hat{E}_i$ and $\hat{E}_{i+1}$. Then

$$K_{E_i} = \pi_i^* K_{\hat{E}_i} + \mathcal{D} \sim -2\gamma_i + (2g - 2 + \gamma_i^2)F_i + \mathcal{D},$$

where F_i is a generic fibre on E_i. In particular $K_{E_i}^2 = 8(1-g) + \mathcal{P}$. Similarly

$$K_{E_{i+1}} \sim -2\gamma_i + (2g - 2 + \gamma_i^2)F_{i+1} + \mathcal{D}.$$

By the adjunction formula we can compute

$$\begin{aligned}(\gamma_i^2)_{E_{i+1}} &= (K_{E_i} - K_Y|_{E_i})E_{i+1}|_{E_i} \\ &= -(\gamma_i^2)_{E_i} + 2(g-1) - K_Y E_i E_{i+1} + \mathcal{P},\end{aligned}$$

where $(\gamma^2)_E$ is the self-intersection of the curve γ computed in the surface $E \supset \gamma$. Similarly

$$(\gamma_i^2)_{E_i} = -(\gamma_i^2)_{E_{i+1}} + 2(g-1) - K_Y E_i E_{i+1} + \mathcal{P}.$$

Together these give

$$E_i^2 E_{i+1} + E_i E_{i+1}^2 = 2(g-1) - K_Y E_i E_{i+1} + \mathcal{P} \quad \text{for } i = 1, \dots, r-2.$$

Next,

$$E_i^2 E_{i+1} + E_{i+1} E_{i+2}^2 = (\gamma_i^2)_{E_{i+1}} + (\gamma_{i+1}^2)_{E_{i+1}} + \mathcal{P} \quad \text{for} \quad i = 1, \dots, r-3$$

since $\pi_{i+1}(\gamma_i)$ and $\pi_{i+1}(\gamma_{i+1})$ are disjoint sections in $\hat{E}_{i+1}$.

By the projection formula the following also holds:

$$K_Y E_{i-1} E_i - K_Y E_i E_{i+1} = \mathcal{P} \text{ and } K_Y E_i^2 + 2K_Y E_i E_{i+1} = \mathcal{P}.$$

Finally, putting together all of the above yields

$$\begin{aligned}E_i^2 E_{i+1} &= -(r-2-i)2(g-1) + (r-2-i)K_Y E_1 E_2 + E_{r-2}^2 E_{r-1} + \mathcal{P}, \\ E_i E_{i+1}^2 &= (r-1-i)2(g-1) - (r-1-i)K_Y E_1 E_2 - E_{r-2}^2 E_{r-1} + \mathcal{P},\end{aligned}$$

for all $i = 1, \dots, r-2$.

Expressions involving the second Chern class $c_2(Y)$ can also be computed without difficulty. Let $\mathcal{N}_i$ be the normal bundle of E_i in the smooth threefold Y, and $\mathcal{T}_{E_i}$, $\mathcal{T}_Y$ the tangent sheaves. There is an exact sequence

$$0 \to \mathcal{T}_{E_i} \to j_i^* \mathcal{T}_Y \to \mathcal{N}_i \to 0,$$

where $j_i\colon E_i \to Y$ is the inclusion. Comparison of Chern polynomials and adjunction gives

$$c_2(Y)\cdot E_i = c_2(E_i) - K_{E_i}E_i|_{E_i} = c_2(E_i) - (K_Y + E_i)E_i^2.$$

Also $c_2(E_i)$ can be expressed with other invariants:

$$\frac{1}{12}\left(K_{E_i}^2 + c_2(E_i)\right) = 1 + p_a(E_i) = 1 - g.$$

This in particular implies that $c_2(E_i) = 4(1-g) + \mathcal{P}$. From

$$8(1-g) + \mathcal{P} = K_{E_i}^2 = (K_Y + E_i)^2 E_i = E_i^3 + 2K_Y E_i^2 + \mathcal{P}$$

we get

$$c_2(Y)\cdot E_i = -\frac{1}{2}E_i^3 + \mathcal{P} = 4(g-1) - 2K_Y E_1 E_2 + \mathcal{P}.$$

This reduces the expression for R_C^3 to

$$\begin{aligned}
&\frac{8(1-g) + 4K_Y E_1 E_2}{r^3}\sum_{i=1}^{r-1}\left(\overline{ki}\right)^3 + \\
&\frac{3}{r^3}\sum_{i=1}^{r-2}(\overline{ik})^2\overline{(i+1)k}\Big(2(r-2-i)(1-g) + (r-2-i)K_Y E_1 E_2 + E_{r-2}^2 E_{r-1}\Big) + \\
&\frac{3}{r^3}\sum_{i=1}^{r-2}\overline{ik}(\overline{(i+1)k})^2\Big(2(r-1-i)(g-1) - (r-1-i)K_Y E_1 E_2 - E_{r-2}^2 E_{r-1}\Big) + \mathcal{P}
\end{aligned}$$

and $R_C\cdot c_2(Y)$ reduces to

$$-\frac{4(1-g) + 2K_Y E_1 E_2}{r}\sum_{i=1}^{r-1}\overline{ki} + \mathcal{P}.$$

Arguing as in [14], we can simplify $R_C^3 + \frac{1}{2}R_C\cdot c_2(Y)$ to the form

$$-\frac{1}{r^2}\overline{k}(r-\overline{k})(r-2\overline{k})\left[(r-2)\left(1-g+\frac{1}{2}K_Y E_1 E_2\right) + E_{r-2}^2 E_{r-1}\right] + \mathcal{P}.$$

This ends the proof of Lemma 2.5. □

Corollary 2.6. *Let (X, D) satisfy the conditions of Theorem 2.1. Then for all positive integers m,*

$$\begin{aligned}
\chi(X, \mathcal{O}_X(mD)) = {} & \chi(\mathcal{O}_X) + \frac{1}{12}mD(mD - K_X)(2mD - K_X) \\
& + m\frac{D\cdot c_2(X)}{12} + \sum_C s_C(mD) + \mathcal{P}_3
\end{aligned}$$

where

$$s_C(mD) = -m\frac{\overline{mk}(r-\overline{mk})}{2r}\deg D|_C + \frac{\overline{mk}(r-\overline{mk})}{4r}\deg K_X|_C +$$

$$\frac{\overline{mk}(r-\overline{mk})(r-2\cdot\overline{mk})}{6r^2}\left((r-2)(1-g)+\frac{r-2}{2}K_Y{}^CE_1\,{}^CE_2+{}^CE_{r-2}^2\,{}^CE_{r-1}\right),$$

and $\mathcal{P}_3$ is a contribution from the dissident and isolated singular points of the threefold (X, D).

Proof. For $m = 1$, we only need to put the results of Lemma 2.4 and Lemma 2.5 into Formula (2.2) on page 608. For $m > 1$, consider $\mathcal{O}_X(mD)$, which is of transverse type $\frac{}{mk}\big(\frac{1}{r}(1,-1)\big)$ on C, and repeat the proof using

$$f^*(mD) = D^{(m)} + \sum_C R_C^{(m)} + \mathcal{D},$$

where $D^{(m)}$ is a Cartier divisor on Y, and

$$R_C^{(m)} = \frac{\overline{mk}}{r}\,{}^CE_1 + \frac{\overline{2mk}}{r}\,{}^CE_2 + \ldots + \frac{\overline{(r-1)mk}}{r}\,{}^CE_{r-1}.$$

□

2.4. The contribution from a dissident or isolated singular point. The analytically invariant contributions to RR from dissident or isolated singular points, denoted by $\mathcal{P}_3$ in Corollary 2.6, can be computed on any model that contains such singularities. We begin by showing the existence of such projective varieties.

Proposition 2.7. *Fix positive integers s and a_1, a_2, a_3, and assume that a_i, a_j, s have no common factor for all different $i, j \in \{1, 2, 3\}$. There exists a smooth projective 3–fold Z together with an action of μ_s with the following properties: the action fixes a number of points on which a generator $\varepsilon \in \mu_s$ acts by*

$$\varepsilon\colon\ z_1, z_2, z_3 \ \mapsto\ \varepsilon^{a_1}z_1, \varepsilon^{a_2}z_2, \varepsilon^{a_3}z_3.$$

If $\mathrm{hcf}(a_i, s) = \alpha_i \neq 1$, *these points lie on curves which are fixed by $\varepsilon^{\frac{s}{\alpha_i}}$. Finally $\varepsilon^{\frac{s}{\alpha_i}} \in \mu_{\alpha_i}$ acts in the normal direction of each curve by*

$$\varepsilon^{\frac{s}{\alpha_i}}\colon\ z_j, z_k \ \mapsto\ \varepsilon^{\frac{s}{\alpha_i}a_j}z_j, \varepsilon^{\frac{s}{\alpha_i}a_k}z_k, \quad \text{for } j,k \in \{1,2,3\} - \{i\}$$

and freely away from the curve.

Proof. We imitate the proof of (8.4) in [21]. Choose an integer $l \geq 3$ and consider the action of μ_s on $\mathbb{P}^{l+3}(1, 1, \ldots, 1)$ given by

$$x_1, x_2, x_3, x_4, \ldots, x_{l+4} \ \mapsto\ \varepsilon^{a_1}x_1, \varepsilon^{a_2}x_2, \varepsilon^{a_3}x_3, x_4, \ldots, x_{l+4}.$$

This action fixes $\mathbb{P}^l = \{x_1 = x_2 = x_3 = 0\}$ and acts in the normal direction by

$$x_1, x_2, x_3 \ \mapsto \ \varepsilon^{a_1} x_1, \varepsilon^{a_2} x_2, \varepsilon^{a_3} x_3.$$

If $\mathrm{hcf}(a_i, s) = \alpha_i \neq 1$, the action is not free on $\mathbb{P}^{l+1} = \{x_j = x_k = 0\}$ for $j, k \in \{1, 2, 3\} - \{i\}$. This is fixed by $\varepsilon^{\frac{s}{\alpha_i}}$ which acts in the normal direction by

$$x_j, x_k \ \mapsto \ \varepsilon^{\frac{s}{\alpha_i} a_j} x_j, \varepsilon^{\frac{s}{\alpha_i} a_k} x_k.$$

Another locus on which the action might not be free is $\{x_4, \ldots, x_{l+4} = 0\}$. We will avoid this locus by defining

$$X \subset \mathbb{P}^{l+3}/\mu_s$$

as a complete intersection of l general very ample divisors. Let Z be the inverse image of X under the quotient $\mathbb{P}^{l+3} \to \mathbb{P}^{l+3}/\mu_s$. Such a Z clearly satisfies the conditions in the proposition. $\square$

Let X be a projective threefold with a singularity of type $\frac{1}{s}(a_1, a_2, a_3)$ as described in Proposition 2.7. We restricted our study to Du Val curves, which means that if $\mathrm{hcf}(a_i, s) = \alpha_i \neq 1$, then

$$a_j + a_k = 0 \mod \alpha_i.$$

Thus $\mathrm{Sing} X$ equals a number of dissident

$$A_{\alpha_i - 1} = \frac{1}{\alpha_i}(a_j, a_k)$$

curves, which intersect in dissident points of type $\frac{1}{s}(a_1, a_2, a_3)$.

Write $\pi \colon Z \to X$ for the quotient map and let $\mathcal{L}_n$ be the nth eigensheaf of the action of $\varepsilon \in \mu_s$ on $\pi_* \mathcal{O}_Z$. Then

$$\pi_* \prime_Z = \bigoplus_{n=0}^{s-1} \mathcal{L}_n$$

implies

$$H^p(Z, \mathcal{O}_Z) = \bigoplus_{n=0}^{s-1} H^p(X, \mathcal{L}_n).$$

The group action on any $f \in \mathcal{L}_n$ is given by $\varepsilon(f) = \varepsilon^n \cdot f$; thus

$$\mathrm{Tr}(\varepsilon : H^p(Z, \mathcal{O}_Z)) = \sum_{n=0}^{s-1} h^p(X, \mathcal{L}_n) \cdot \varepsilon^n.$$

Moreover,

$$\sum_p (-1)^p \mathrm{Tr}(\varepsilon : H^p(Z, \mathcal{O}_Z)) = \sum_{n=0}^{s-1} \chi(X, \mathcal{L}_n) \cdot \varepsilon^n.$$

In order to simplify the notation, denote $\sum_p (-1)^p \mathrm{Tr}(\varepsilon : H^p(Z, \mathcal{O}_Z))$ by A_ε. Then

$$\sum_{n=0}^{s-1} \chi(X, \mathcal{L}_n) \cdot \varepsilon^n = A_\varepsilon \quad \text{and} \quad \sum_{n=0}^{s-1} \chi(X, \mathcal{L}_n) = \chi(\mathcal{O}_Z).$$

The last two formulas can be considered as a linear system of s equations in $\chi(X, \mathcal{L}_n)$ and variable ε,

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \varepsilon^2 & \dots & \varepsilon^{s-1} \\ 1 & \varepsilon^2 & \varepsilon^4 & \dots & \varepsilon^{2(s-1)} \\ \dots & \dots & \dots & \dots & \\ 1 & \varepsilon^{s-1} & \dots & \dots & \varepsilon \end{pmatrix} \begin{pmatrix} \chi(X, \mathcal{L}_0) \\ \chi(X, \mathcal{L}_1) \\ \chi(X, \mathcal{L}_2) \\ \vdots \\ \chi(X, \mathcal{L}_{s-1}) \end{pmatrix} = \begin{pmatrix} \chi(\mathcal{O}_Z) \\ A_\varepsilon \\ A_{\varepsilon^2} \\ \vdots \\ A_{\varepsilon^{s-1}} \end{pmatrix}.$$

Write $\chi(X, \mathcal{L}_0) = \chi(\mathcal{O}_X)$ and eliminate $\chi(\mathcal{O}_Z)$ from the solution. We end up with

$$\begin{aligned} \chi(X, \mathcal{L}_n) &= \chi(\mathcal{O}_X) + \frac{1}{s} \sum_{j=1}^{s-1} (\varepsilon^{-jn} - 1) A_{\varepsilon^j} \\ &= \chi(\mathcal{O}_X) + \frac{1}{s} \sum_{\varepsilon \in \mu_s} (\varepsilon^{-n} - 1) A_\varepsilon. \end{aligned} \tag{2.3}$$

We can compute A_ε for all $\varepsilon \in \mu_s$ using the Atiyah–Singer–Segal equivariant RR formula [3], [4, p.565]. If $\varepsilon^{a_i} \neq 1$ for all $i = 1, 2, 3$, then the fixed locus of ε is a number of points. In this case, for each point

$$A_\varepsilon = \sum_p (-1)^p \mathrm{Trace}(\varepsilon | H^p(Z, \mathcal{O}_Z)) = \frac{1}{(1 - \varepsilon^{-a_1})(1 - \varepsilon^{-a_2})(1 - \varepsilon^{-a_3})}.$$

If, on the other hand, $\varepsilon^{a_i} = 1$ for one of the $i = 1, 2, 3$, then the fixed locus of ε is a curve C_i. In this case, $\mathrm{hcf}(a_i, s) = \alpha_i \neq 1$ and ε is an element of μ_{α_i}. The action in the normal direction equals

$$\varepsilon^{a_j}, \varepsilon^{a_k} = \varepsilon^{a_j}, \varepsilon^{-a_j}.$$

Thus, we can use the equivariant Riemann–Roch formula for the cyclic group μ_{α_i}. Let C_i have genus g_i, and let x_1, x_2 denote the first Chern classes of $\mathcal{N}_{C_i/Z} = \mathcal{N}_1 \oplus \mathcal{N}_2$. We get

$$\begin{aligned} A_\varepsilon &= \textstyle\sum_p (-1)^p \mathrm{Trace}(\varepsilon | H^p(Z, \mathcal{O}_Z)) \\ &= \deg \left\{ \frac{\mathrm{td}(\mathcal{T}_C)}{(1 - \varepsilon^{-a} e^{-x_1})(1 - \varepsilon^{-(r-a)} e^{-x_2})} \right\}_1 \\ &= \tfrac{1 - g_i}{(1 - \varepsilon^{-a_j})(1 - \varepsilon^{a_j})} - \tfrac{\varepsilon^{-a_j}}{(1 - \varepsilon^{-a_j})^2 (1 - \varepsilon^{a_j})} \deg_{C_i} x_1 - \tfrac{\varepsilon^{a_j}}{(1 - \varepsilon^{-a_j})(1 - \varepsilon^{a_j})^2} \deg_{C_i} x_2. \end{aligned}$$

Since we assumed that a_i, a_j, s have no common divisor for all $i, j = 1, 2, 3$, we can split the sum over $\{\varepsilon \in \mu_s\}$ into four subsums over (possibly empty) disjoint sets

$$\{\varepsilon \in \mu_s \text{ s.t. } \varepsilon^{a_i} \neq 1 \ \forall i = 1, 2, 3\},$$

$$\{\varepsilon \in \mu_s \text{ s.t. } \varepsilon^{a_1} = 1\} = \{\varepsilon \in \mu_{\alpha_1}\},$$

$$\{\varepsilon \in \mu_s \text{ s.t. } \varepsilon^{a_2} = 1\} = \{\varepsilon \in \mu_{\alpha_2}\} \text{ and}$$

$$\{\varepsilon \in \mu_s \text{ s.t. } \varepsilon^{a_3} = 1\} = \{\varepsilon \in \mu_{\alpha_3}\}.$$

This rewrites Formula (2.3) into

$$\chi(X, \mathcal{L}_n) = \chi(\mathcal{O}_X) + \frac{1}{s} \sum_{\substack{\varepsilon \in \mu_s \\ \varepsilon^{a_i} \neq 1 \ \forall i = 1,2,3}} \frac{\varepsilon^{-n} - 1}{(1 - \varepsilon^{-a_1})(1 - \varepsilon^{-a_2})(1 - \varepsilon^{-a_3})}$$

$$+ \frac{1}{s} \sum_{(i,j)} \sum_{\varepsilon \in \mu_{\alpha_i}} (\varepsilon^{-n} - 1) \left(\frac{1 - g_i}{(1 - \varepsilon^{-a_j})(1 - \varepsilon^{a_j})} - \frac{\varepsilon^{-a_j}}{(1 - \varepsilon^{-a_j})^2 (1 - \varepsilon^{a_j})} \deg_{C_i} x_1 \right.$$
$$\left. - \frac{\varepsilon^{a_j}}{(1 - \varepsilon^{-a_j})(1 - \varepsilon^{a_j})^2} \deg_{C_i} x_2 \right),$$

where $(i, j) \in \{(1, 2),\ (2, 3),\ (3, 1)\}$.

Since $c_1(\mathcal{N}_{C_i/Z}) = K_{C_i} - K_Z|_{C_i}$, we get $\deg_{C_i} x_1 + \deg_{C_i} x_2 = 2(g_i - 1) - \deg K_Z|_{C_i}$. Then

$$\chi(X, \mathcal{L}_n) = \chi(\mathcal{O}_X) + \frac{1}{s} \sum_{\substack{\varepsilon \in \mu_s \\ \varepsilon^{a_i} \neq 1 \ \forall i = 1,2,3}} \frac{\varepsilon^{-n} - 1}{(1 - \varepsilon^{-a_1})(1 - \varepsilon^{-a_2})(1 - \varepsilon^{-a_3})}$$

$$+ \sum_{i=1}^{3} \frac{\overline{nk_i}(\alpha_i - \overline{nk_i})}{4s} \deg K_Z|_{C_i}$$

$$+ \frac{1}{12s} \overline{nk_i}(\alpha_i - \overline{nk_i})(\alpha_i - 2 \cdot \overline{nk_i})(\deg_{C_i} x_1 - \deg_{C_i} x_2),$$

where $\overline{}$ denotes the smallest residue mod α_i and the k_i are defined by $k_i a_j = 1$ mod α_i for (i, j) as above.

We can think of $\chi(\mathcal{O}_X)$ as the RR-type expression for the pair (X, L_n), where $L_n = 0 \in \operatorname{Div} X \otimes \mathbb{Q}$ is the $\mathbb{Q}$–divisor corresponding to $\mathcal{L}_n$. The rest of the formula is a sum of contributions coming from

- dissident points of type $\frac{1}{s}(a_1, a_2, a_3)$, and
- dissident curves of type $\frac{1}{\alpha_i}(a_j, -a_j)$,

which are the singularities of (X, L_n).

Remark 2.8. The attentive reader will notice that the argument in the last two sections is completely different: for A_r curves, we use resolution of singularities and computations on ruled surfaces, whereas for (isolated or dissident) singular points, we use an auxiliary cover and equivariant Riemann–Roch. At least for curves without dissident points, the auxiliary cover construction also works, though with a twist. For an A_r curve, a cyclic cover may not exist, but a $\mu_r \oplus \mu_r$ cover necessarily does and the argument goes through. The invariant N_C of X (the only term not explicitly defined in Theorem 2.1) depends only on the first-order neighbourhood of the curve C and can be expressed explicitly in terms of the genus of C, the singularity type of X along C and splitting of the normal bundle of the curve $\widetilde{C}$ over C in the auxiliary cover. For details, consult [8].

2.5. Conclusion of the proof.

Proof of Theorem 2.1. By Corollary 2.6, we only need to add the contributions from the dissident and isolated singular points to the RR formula. These were computed above.

Let C be a $\frac{}{mk}\big(\frac{1}{r}(1,-1)\big)$ curve for $\mathcal{O}_X(mD)$ with dissident points

$$\{P_\lambda \mid \lambda \in \Lambda\} \text{ of types } \tfrac{}{mn_\lambda}\Big(\frac{1}{s_\lambda}(a_{1\lambda}, a_{2\lambda}, a_{3\lambda})\Big).$$

Note that for every λ there exists i such that $\operatorname{hcf}(a_{i\lambda}, s_\lambda) = r$, since $P_\lambda \in C$. Moreover, $mn_\lambda a_{j\lambda}^{-1} = mk \bmod r$.

The contribution to RR from this locus is a sum of

$$\frac{\overline{mk}(r-\overline{mk})(r-2\cdot\overline{mk})}{6r^2}\left((r-2)(1-g) + \tfrac{r-2}{2}K_Y\, {}^C E_1\, {}^C E_2 + {}^C E_{r-2}^2\, {}^C E_{r-1}\right)$$

by Corollary 2.6, and of

$$\tfrac{1}{12s_\lambda}\overline{mk}(r-\overline{mk})(r-2\cdot\overline{mk})(\deg_{C_{i\lambda}} x_1 - \deg_{C_{i\lambda}} x_2)$$
$$+\frac{1}{s_\lambda}\sum_{\substack{\varepsilon\in\mu_{s_\lambda}\\ \varepsilon^{a_{i\lambda}}\neq 1\,\forall i=1,2,3}} \frac{\varepsilon^{-mn_\lambda}-1}{(1-\varepsilon^{-a_{1\lambda}})(1-\varepsilon^{-a_{2\lambda}})(1-\varepsilon^{-a_{3\lambda}})}$$

for every point P_λ. The first two rows of the sum can be written together as

$$\frac{\overline{mk}(r-\overline{mk})(r-2\cdot\overline{mk})}{12r^2\tau_C}N_C,$$

where $\tau_C = \frac{1}{r}\operatorname{lcm}\{s_\lambda\}$ is the index of C and the integer N_C is an invariant of X in the neighbourhood of C. $\square$

3. Calabi–Yau threefolds

3.1. The Hilbert series. When X is a Calabi–Yau threefold, the RR formula assumes a much more compact form.

Corollary 3.1. *Let X be a Calabi–Yau threefold and let D be a $\mathbb{Q}$-Cartier divisor. Assume that (X, D) is quasi-smooth and well formed with the following singularities:*

- *points Q of type ${}_n\big(\frac{1}{s}(a_1, a_2, a_3)\big)$,*
- *curves C of generic type ${}_k\big(\frac{1}{r}(1, -1)\big)$ with index τ_C.*

Then for all positive integers m,

$$h^0(X, \mathcal{O}_X(mD)) = \frac{1}{6}m^3D^3 + m\frac{D \cdot c_2(X)}{12} + \textstyle\sum_Q c_Q(mD) + \sum_C s_C(mD),$$

where

$$c_Q(mD) = \frac{1}{s} \sum_{\substack{\varepsilon \in \mu_s \\ \varepsilon^{a_i} \neq 1\, \forall i=1,2,3}} \frac{\varepsilon^{-nm} - 1}{(1 - \varepsilon^{-a_1})(1 - \varepsilon^{-a_2})(1 - \varepsilon^{-a_3})}$$

and

$$s_C(mD) = -m\frac{\overline{mk}(r - \overline{mk})}{2r} \deg D|_C + \frac{\overline{mk}(r - \overline{mk})(r - 2 \cdot \overline{mk})}{12r^2\tau_C} N_C,$$

where $\overline{}$ denotes the smallest residue mod r and the integer N_C is an invariant of X in the neighbourhood of C.

Proof. This is a direct corollary of Theorem 2.1. By Kodaira vanishing, the higher cohomologies of $\mathcal{O}_X(mD)$ vanish for $m > 0$, and $\chi(\mathcal{O}_X) = 0$ by Serre duality. □

Remark 3.2. Observe that for a $\frac{1}{2}(1, 1)$ curve of singularities, the term involving N_C vanishes. The Riemann–Roch formula for this special case was already proved in [22].

Corollary 3.3. *If (X, D) satisfies the conditions of Corollary 3.1, then the Hilbert series*

$$P_X(t) = 1 + \sum_{m=1}^{\infty} h^0(X, \mathcal{O}_X(mD))t^m$$

can be written in the compact form

$$P_X(t) = 1 + \frac{D^3}{6} \cdot \frac{t^3 + 4t^2 + t}{(1 - t)^4} + \frac{c_2(X) \cdot D}{12} \cdot \frac{t}{(1 - t)^2} + \textstyle\sum_Q \tilde{P}_Q(t) + \sum_C \tilde{P}_C(t),$$

where every curve C contributes

$$\tilde{P}_C(t) = -\deg D|_C \left(\frac{1}{1-t^r} \sum_{i=1}^{r-1} i \frac{\overline{ik}(r-\overline{ik})}{2r} t^i + \frac{rt^r}{(1-t^r)^2} \sum_{i=1}^{r-1} \frac{\overline{ik}(r-\overline{ik})}{2r} t^i \right)$$

$$+ \frac{N_C}{12r^2\tau_c} \cdot \frac{1}{1-t^r} \sum_{i=1}^{r-1} \overline{ik}(r-\overline{ik})(r-2\cdot\overline{ik})t^i$$

and every singular point Q contributes

$$\tilde{P}_Q(t) = \frac{1}{1-t^s} \sum_{i=1}^{s-1} c_Q(iD)t^i.$$

Here

$$c_Q(iD) = \frac{1}{s} \sum_{\substack{\varepsilon\in\mu_s \\ \varepsilon^{a_i}\neq 1\ \forall i=1,2,3}} \frac{\varepsilon^{-ni}-1}{(1-\varepsilon^{-a_1})(1-\varepsilon^{-a_2})(1-\varepsilon^{-a_3})}.$$

Proof. This follows directly from Corollary 3.1, using elementary summation formulae of power series. □

3.2. Examples. We apply Corollary 3.3 to the construction of new families of projective Calabi–Yau threefolds as follows. In the first step, we compute the Hilbert series $P(t)$ from Corollary 3.3 using the following input data:

- integers $h^0(X,D)$ and $h^0(X,2D)$;
- points $\left\{{}_n\left(\frac{1}{s}(a_1,a_2,a_3)\right)\right\}$;
- curves $\left\{{}_k\left(\frac{1}{r}(1,-1)\right)\right.$ of degree $\deg D|_C$, index τ_C, invariant $\left.N_C\right\}$.

In the next step, we look for a set of weights $w_0,\dots,w_n$ such that

$$Q(t) = P(t)\prod_{k=0}^{n}(1-t^{w_k})$$

is a polynomial. Then a plausible guess is that (X,D) can be embedded in the ambient space $\mathbb{P}(w_0,\dots,w_n)$, and the shape of $Q(t)$ will suggest a set of generators and relations for the defining ideal. Compare [1, 2] for the philosophy and some explicit examples of this type of argument. As discussed in [2] in great detail, this procedure is best done by a computer; we thank Gavin Brown for providing a computer program written in the programming language of the Magma computer algebra system [7].

Lists of complete intersection Calabi–Yau threefolds in weighted projective spaces can certainly be generated in this way; such Calabi–Yau manifolds have been listed by direct methods and extensively studied in the literature, for example in [9, 10, 17]. Our computer search also generates more interesting examples in higher codimensions.

Example 3.4. Take the input data

- $h^0(X, D) = 3$ and $h^0(X, 2D) = 6$;
- points $\{{}_2\big(\frac{1}{3}(1,1,1)\big), {}_8\big(\frac{1}{9}(1,3,5)\big)\}$;
- curve $\frac{1}{3}(1,2)$ with $\deg D|_C = \frac{1}{9},\ \tau_C = 3,\ N_C = 22$.

The Hilbert series is

$$P(t) = \frac{Q(t)}{(1-t)^3(1-t^3)^2(1-t^5)(1-t^9)},$$

where

$$Q(t) = -t^{23} + t^{17} + t^{15} + 2t^{13} + t^{11} - t^{12} - 2t^{10} - t^8 - t^6 + 1.$$

This suggests that (X, D) could be realized as a codimension 3 threefold in the weighted projective space $\mathbb{P}^6(1^3, 3^2, 5, 9)$. Indeed, define $X \subset \mathbb{P}^6(1^3, 3^2, 5, 9)$ by the vanishing of the submaximal Pfaffians

$$\begin{aligned}
\mathrm{Pf}_1 &= b_7 y_1 + b_5 z + x_2 v,\\
\mathrm{Pf}_2 &= a_7 y_1 + a_5 z + x_1 v,\\
\mathrm{Pf}_3 &= y_2 y_1 - a_5 x_2 + b_5 x_1,\\
\mathrm{Pf}_4 &= -y_2 z - a_7 x_2 + b_7 x_1,\\
\mathrm{Pf}_5 &= y_2 v - a_7 b_5 + a_5 b_7
\end{aligned}$$

of the 5×5 skew-symmetric matrix

$$M = \begin{pmatrix} 0 & y_2 & a_7 & a_5 & x_1 \\ & 0 & b_7 & b_5 & x_2 \\ & & 0 & v & -z \\ & & & 0 & y_1 \\ & & & & 0 \end{pmatrix}.$$

Here $x_1, x_2, x_3, y_1, y_2, z, v$ are weighted variables of $\mathbb{P}^6$ and a_i, b_i are general polynomials of degree i. It is easy to check that X is quasi-smooth and well formed, and that its singular locus indeed consists of an isolated singular point of type $\frac{1}{3}(1,1,1)$ and an A_2 curve with a dissident point of type $\frac{1}{9}(1,3,5)$.

From this representation, one can also prove that X is a general quasilinear section

$$X = (5)^2 \cap (7)^2 \cap \mathcal{C}\mathrm{wGr}(2,5) \subset \mathbb{P}(1^3, 3^2, 5^3, 7^2, 9).$$

Here $\mathcal{C}\mathrm{wGr}(2,5)$ denotes the projective cone over $\mathrm{wGr}(2,5)$, the weighted Grassmannian as defined in [11].

Example 3.5. Take the input data

- $h^0(X, D) = 2$ and $h^0(X, 2D) = 4$;
- point ${}_4\big(\frac{1}{5}(1,1,3)\big)$;
- curve $\frac{1}{3}(1,2)$ with $\deg D|_C = 1,\ \tau_C = 1,\ N_C = 12$.

The computer output for $P(t)$ is

$$P(t) = \frac{1 - 3t^6 - 3t^7 - 3t^8 + 2t^9 + 6t^{10} + 6t^{11} + \ldots + t^{21}}{(1-t)^2(1-t^2)(1-t^3)^4(1-t^5)}.$$

This suggests that (X, D) could be realized as a codimension 4 threefold in the weighted projective space $\mathbb{P}^7(1^2, 2, 3^4, 5)$, defined by nine relations of degrees $6, 6, 6,\ 7, 7, 7,\ 8, 8, 8$ respectively. The existence of X can be proved by Type I unprojection [1, 18] starting from a codimension 3 variety $Y \subset \mathbb{P}^6(1, 1, 2, 3, 3, 3, 3)$ defined by a set of Pfaffians as in Example 3.4. We omit the details.

Example 3.6. Take the input data

- $h^0(X, D) = 2$ and $h^0(X, 2D) = 7$;
- point ${}_3\big(\frac{1}{4}(2, 3, 3)\big)$;
- curve $\frac{1}{2}(1, 1)$ with $\deg D|_C = \frac{7}{4},\ \tau_C = 2$.

The closed form of the Hilbert series is

$$P(t) = \frac{1 - 3t^4 - 4t^5 - t^6 + 6t^7 + 6t^8 + 2t^9 - 2t^{11} - \ldots - t^{20}}{(1-t)^2(1-t^2)^4(1-t^3)^2(1-t^4)}.$$

This suggests that (X, D) could be realized as a codimension 5 threefold in the weighted projective space $\mathbb{P}^8(1^2, 2^4, 3^2, 4)$. This turns out to be true indeed, although the details are somewhat tedious. More conceptually, following [11], one can show that (X, D) can be realized as a general quasilinear section

$$X = \mathrm{wOGr}(5, 10) \cap (2)^2 \cap (3)^4 \cap (4) \subset \mathbb{P}(1^2, 2^6, 3^6, 4^2),$$

where $\mathrm{wOGr}(5, 10)$ denotes a weighted orthogonal Grassmannian [11].

Remark 3.7. Two questions arise at this point: to what extent are these Calabi–Yau threefold families "new", and how many families can one construct this way. The second question will be discussed elsewhere; to give away the (negative) punchline, however hard we try, so far we have been unable to construct infinitely many families. In answer to the first question, it is certainly possible that our examples are birational to one of the gigantic number of toric complete intersection Calabi–Yau threefolds [5, 6, 16]. However, our projective descriptions, in relatively simple ambient varieties such as weighted Grassmannians [11] and the "universal Type I unprojection" [18], are new. Such descriptions are quite pretty in themselves and can also be used for various purposes such as computing their Hodge numbers and studying mirror symmetry phenomena (details to follow). For these reasons alone, we believe they deserve some attention.

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