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Sheaves on Fibered Threefolds and Quiver Sheaves

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Abstract: This paper classifies a class of holomorphic D-branes, closely related to framed torsion-free sheaves, on threefolds fibered in resolved ADE surfaces over a general curve C, in terms of representations with relations of a twisted Kronheimer–Nakajima-type quiver in the category $\mathrm{Coh}(C)$ of coherent sheaves on C. For the local Calabi–Yau case $C \cong \mathbf{A}^1$ and special choice of framing, one recovers the $N = 1$ ADE quiver studied by Cachazo–Katz–Vafa.

Introduction

The purpose of this paper is to study, via dimensional reduction, certain holomorphic D-branes, closely related to torsion-free sheaves, on threefolds $X \to C$ fibered in resolved ADE surfaces over a curve. Fibered local Calabi–Yau threefolds $X \to \mathbb{A}^1$ of this type, as well as their deformations $X_s \to \mathbb{A}^1$ and extremal transitions, were thoroughly analyzed in [6,7] from the point of view of supersymmetric gauge theory. The paper [6] contains an assertion, made explicit in [14] and studied in [21], that exceptional components of a natural threefold contraction $X_s \to \bar{X}_s$ are classified by irreducible representations of a certain quiver with loop edges, the $N = 1$ ADE quiver (see Fig. 3.2 for an example), satisfying a specific set of relations. This statement is in the spirit of Gabriel's theorem classifying exceptional (not necessarily irreducible) rational curves in resolved ADE surfaces in terms of irreducible representations of the corresponding Dynkin quiver.

In this paper we generalize the work of [6,14,21] in two directions: we consider holomorphic D-branes, objects in the derived category of coherent sheaves, instead of exceptional components, and we study the semi-local case: the neighbourhood of a deformed ADE fibration $X_s \to C$ over a general curve C. The main result is Theorem 3.1, which shows that certain holomorphic D-branes on the fibered threefold X_s are classified by representations with relations of a Kronheimer–Nakajima-type quiver

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in the category Coh(C) of coherent sheaves on the curve C. In particular, moduli spaces of such holomorphic D-branes are quiver bundle varieties over C. If $C \cong \mathbb{A}^1$, a further dimensional reduction leads to Theorem 3.4, relating sheaves on the threefold to the zero-dimensional problem of ordinary matrix representations of the $N = 1$ ADE quiver of [6,14,21]. The loops in the $N = 1$ ADE quiver arise as the action by multiplication of a parameter $t \in H^0(\mathcal{O}_{\mathbb{A}^1})$ on spaces of sections of sheaves on the base $\mathbb{A}^1$.

The geometry considered in this paper is non-monodromic, meaning that there is no global [19] nor local [6] monodromy in the fibration of ADE surfaces over the curve C. It appears to be an interesting question to extend the results proved here to these more general cases involving monodromy.

In recent work [9], the moduli space of certain very special holomorphic D-branes on resolved A_1-fibered geometries $X \to C$ has been connected, via imposing a superpotential and going through a large N transition, to the Hitchin system on C. The branes studied in [9] are not of the type classified by our results; they should rather correspond to a complex of quiver representations. Understanding the precise connection between [9] and the present paper is left for future work.

After introducing basic notation in Sect. 1, Sect. 2 describes the threefolds we study, and defines some auxiliary sheaves of non-commutative algebras over the curve C. Section 3 contains our results, in particular the general statement Theorem 3.1 connecting quiver bundles to holomorphic D-branes on ADE fibrations, as well as the statement for the affine case. Proofs are discussed in Sect. 4.

1. Finite Groups of Type ADE and Surfaces

Let $\Gamma < \mathrm{SL}(2,\mathbb{C})$ be a finite subgroup of type A, D or E. Let $\mathfrak{h}_0$ be the Cartan subalgebra of the finite dimensional Lie algebra of the same type. Fix a set of simple roots $\{\eta_a : a \in \Delta_0\}$ indexed by nodes of the Dynkin diagram Δ_0, and let R_+ be the set of positive roots. Let $\mathfrak{h}$ be the corresponding affine Cartan with simple roots indexed by nodes of the Dynkin diagram $\Delta \supset \Delta_0$.

The group ring $\mathbb{C}\Gamma$ has center $Z(\mathbb{C}\Gamma) \cong \mathbb{C}^\Delta$; explicitly, for $\lambda \in Z(\mathbb{C}\Gamma)$, the isomorphism is obtained by taking the trace of λ on a set of irreps, indexed by the nodes of Δ according to the McKay correspondence. There is also a natural identification

$$\mathfrak{h}_0 = \{\lambda \in \mathbb{C}^\Delta \mid \lambda \cdot \delta = 0\} \subset \mathfrak{h} \cong \mathbb{C}^\Delta,$$

where $\delta = (\delta_a)$ are the dimensions of the irreps of Γ.

Lemma 1.1. *The centralizer* $C_{\mathrm{GL}(2,\mathbb{C})}(\Gamma)$ *of* Γ *in* $\mathrm{GL}(2,\mathbb{C})$ *is*

(1) *the full group* $\mathrm{GL}(2,\mathbb{C})$ *for type* A_1*;*
(2) *a torus* $(\mathbb{C}^\star)^2$ *in* $\mathrm{GL}(2,\mathbb{C})$ *for type* A_n *with* $n > 1$*;*
(3) *the center* $\mathbb{C}^\star$ *of* $\mathrm{GL}(2,\mathbb{C})$ *for types* D *and* E*.*

Let $\bar{Y} = \mathbb{A}^2/\Gamma$ be the singular affine quotient, $Y \to \bar{Y}$ its minimal resolution. Exceptional curves in the resolution are in one-to-one correspondence with the nodes of Δ_0, and thus with a set of simple roots of $\mathfrak{h}_0$; the positive roots $\eta \in R_+$ correspond to connected, possibly reducible exceptional rational curves. The universal deformations $\mathcal{Y} \to \mathfrak{h}_0$ and $\bar{\mathcal{Y}} \to \mathfrak{h}_0/W$ of Y and $\bar{Y}$, where W denotes the Weyl group, are connected by the well known commutative diagram

$$\begin{array}{ccccc} \mathcal{Y} & \longrightarrow & p^*\bar{\mathcal{Y}} & \longrightarrow & \bar{\mathcal{Y}} \\ & \searrow & \downarrow & & \downarrow \\ & & \mathfrak{h}_0 & \xrightarrow{p} & \mathfrak{h}_0/W. \end{array}$$

2. Threefolds: Definitions

2.1. The geometry. Let C be a curve, and let $\mathcal{Q}$ be a rank-two vector bundle on C whose structure group reduces from $\mathrm{GL}(2, \mathbb{C})$ to the centralizer $C_{\mathrm{GL}(2,\mathbb{C})}(\Gamma)$. Thus, by Lemma 1.1,

- for type A_1, $\mathcal{Q}$ is an arbitrary rank-two vector bundle;
- for type A_n with $n > 1$, $\mathcal{Q} \cong \mathcal{Q}_1 \oplus \mathcal{Q}_2$ is the direct sum of two line bundles;
- for types D, E, $\mathcal{Q} \cong \mathcal{Q}_0^{\oplus 2}$ for some line bundle $\mathcal{Q}_0$.

There is a fiberwise Γ-action on the total space of the vector bundle $\mathcal{Q}$, and the quotient $\bar{X} = \mathcal{Q}/\Gamma$ is a threefold with a curve of compound Du Val singularities along the image of the zero section. Let $f : X \to \bar{X}$ be the crepant resolution, with a map $\pi : X \to C$ whose fibres are minimal resolutions of the corresponding surface singularity, with trivial monodromy in the fibres. The canonical bundle of X is

$$\omega_X \cong \pi^*(\omega_C \otimes \det \mathcal{Q}^\vee).$$

In particular, X is Calabi–Yau if and only if $\mathcal{Q}$ has canonical determinant on C.

Part of the deformation theory of the threefold X was described in [19]. Let $\mathcal{H}_0 = \det \mathcal{Q} \otimes \mathfrak{h}_0$, a vector bundle over C, and let $\mathcal{S} = H^0(C, \mathcal{H}_0)$ be its space of sections. Then there is a smooth family of threefolds $\mathcal{X} \to \mathcal{S}$, with injective Kodaira–Spencer map and central fibre $X_0 \cong X$, together with a fibration $\mathcal{X} \to C \times S$ and a contraction $\mathcal{X} \to \bar{\mathcal{X}}$ over S. Thus, for every $s \in \mathcal{S}$, the threefold fibre X_s possesses a fibration $\pi_s : X_s \to C$ in surfaces and a contraction $f_s : X_s \to \bar{X}_s$ to a singular threefold with compound Du Val singularities. More precisely, for every positive root $\eta \in R_+$ of $\mathfrak{h}_0$, there is a map $p_\eta : \mathcal{H}_0 \to \det \mathcal{Q}$, whose vanishing locus is a family of root hyperplanes in the $\mathfrak{h}_0$ fibers, and we have

Lemma 2.1. *Let $s \in \mathcal{S} = H^0(C, \mathcal{H}_0)$ be a section of $\mathcal{H}_0$, and let $\eta \in R_+$ be a positive root of $\mathfrak{h}_0$. The contraction $f_s : X_s \to \bar{X}_s$ contracts a (possibly reducible) rational curve corresponding to the root η over a point $P \in C$, if and only if the projected section $p_\eta(s) \in H^0(C, \det \mathcal{Q})$ vanishes at $P \in C$.*

Thus if the projected section $p_\eta(s)$ is not identically zero for any root η, then f_s is a small contraction, contracting rational curves to isolated singularities in certain configurations. If for different roots η, the sections $p_\eta(s)$ have different simple zeros, then f_s contracts a set of isolated $(-1, -1)$-curves to simple nodes. If the linear system $\det \mathcal{Q}$ has no base points on C, then this holds for generic $s \in \mathcal{S}$.

In the special case $C \cong \mathbb{A}^1$, the central fiber $X_0 = \mathbb{A}^1 \times Y$ is Calabi–Yau, and its deformations are parameterized by an $\mathfrak{h}_0$-valued polynomial $s \in \mathfrak{h}_0[t]$. Under the isomorphism $\mathfrak{h}_0 \cong \{\lambda \mid s \cdot \delta = 0\} \subset \mathbb{C}^\Delta$, we can also parameterize deformations by a set of ordinary polynomials $\Theta_a \in \mathbb{C}[t]$ indexed by nodes of the affine Dynkin diagram Δ, satisfying $\sum_a \delta_a \Theta_a = 0$. The exceptional fibres of $f_s : X_s \to \bar{X}_s$ lie over roots of the various polynomials $\Theta_{\eta_a} = \Theta_a$, corresponding to simple roots η_a, as well as over roots of their linear combinations $\Theta_\eta = \sum_a \mu_a \Theta_a$, corresponding to other positive roots $\eta = \sum_a \mu_a \eta_a \in R_+$. For generic choice of parameter $s \in \mathcal{S}$, equivalently for generic choice of $\{\Theta_a\}$, the polynomials $\{\Theta_\eta : \eta \in R_+\}$ have distinct simple roots, and the exceptional set of $f_s : X_s \to \bar{X}_s$ consists of isolated $(-1, -1)$-curves.

2.2. Sheaves of non-commutative algebras and their sheaves of modules. Given $(C, \mathcal{Q})$, let $\mathcal{H} = \det \mathcal{Q} \otimes \mathfrak{h}$, a vector bundle on the curve C containing $\mathcal{H}_0$ as a subbundle. Given a section $s \in H^0(C, \mathcal{H})$, consider the natural composition

$$\sigma_s : \quad \mathcal{Q}^\vee \otimes \mathcal{Q}^\vee \xrightarrow{\wedge^2} \det \mathcal{Q}^\vee \xrightarrow{\cdot s} \mathfrak{h} \otimes \mathcal{O}_C \xrightarrow{\sim} Z(\mathbb{C}\Gamma) \otimes \mathcal{O}_C,$$

a family of $Z(\mathbb{C}\Gamma)$-valued symplectic forms in the fibres of the vector bundle $\mathcal{Q}^\vee$. Also fix, once and for all, a trivializing section $z \in H^0(\mathcal{O}_C)$.

Definition 2.2. *Let $\mathcal{A}_s$ be the sheaf of non-commutative algebras on C whose sections on an open set $U \subset C$ are*

$$\mathcal{A}_s(U) = T\mathcal{Q}^\vee(U) * \mathbb{C}\Gamma \big/ \langle\langle [x_1, x_2] + \sigma_s(x_1, x_2) \rangle\rangle ,$$

where $T\mathcal{Q}^\vee(U)$ is the full tensor algebra of $\mathcal{Q}^\vee(U)$, $x_i \in \mathcal{Q}^\vee(U)$ are local sections, and $\langle\langle \dots \rangle\rangle$ denotes the two-sided ideal generated by all given expressions. Define also

$$\mathcal{P}_s(U) = T(\mathcal{Q}^\vee \oplus \mathcal{O}_C)(U) * \mathbb{C}\Gamma \Big/ \left\langle\!\!\left\langle [x_1, x_2] + \sigma_s(x_1, x_2) z^2, [x_i, z] \right\rangle\!\!\right\rangle,$$

where the fixed section $z \in H^0(\mathcal{O}_C)$ commutes with elements of $\mathbb{C}\Gamma$. The sheaf $\mathcal{P}_s$ becomes a sheaf of graded algebras by assigning degree 1 to local sections $x_i \in \mathcal{Q}^\vee(U)$ as well as to $z \in H^0(\mathcal{O}_C)$; thus its degree-zero piece is

$$\mathcal{P}_{s,0} \cong \mathcal{O}_C \otimes \mathbb{C}\Gamma.$$

Remark 2.3. The sheaf of algebras $\mathcal{A}_s$ is a relavitive version of the following non-commutative deformation of the skew group algebra, introduced by Crawley–Boevey and Holland in [8], depending on a deformation parameter $\lambda \in \mathfrak{h} \cong Z(\mathbb{C}\Gamma)$:

$$A_\lambda = \mathbb{C}\langle x_1, x_2 \rangle * \Gamma \big/ \langle\langle [x_1, x_2] + \lambda \rangle\rangle.$$

The graded version is

$$P_\lambda = \mathbb{C}\langle y_0, y_1, y_2 \rangle * \Gamma \Big/ \left\langle\!\!\left\langle [y_0, y_i], [y_1, y_2] + \lambda y_0^2 \right\rangle\!\!\right\rangle.$$

For $\Gamma = \{1\}$, λ is just a complex number; if $\lambda \neq 0$, A_λ is isomorphic to the first Weyl algebra, whereas P_λ is a degenerate Sklyanin algebra deforming the algebra of functions on the commutative projective plane $\mathbb{P}^2$. As proved in [8], for general Γ and $\lambda \in \mathfrak{h}_0 \subset Z(\mathbb{C}\Gamma)$ the algebra A_λ is finite over its center

$$ZA_\lambda \cong \mathbb{C}[\bar{Y}_\lambda].$$

The latter is the coordinate ring of the affine variety $\bar{Y}_\lambda$ corresponding to the deformation parameter $\lambda \in \mathfrak{h}_0$, a deformation of the invariant ring $\mathbb{C}[x_1, x_2]^\Gamma \cong \mathbb{C}[\bar{Y}]$. For $\lambda \in \mathfrak{h} \setminus \mathfrak{h}_0$, A_λ is "genuinely" non-commutative.

By abuse of notation, we will refer to $\mathbf{P}_s = \mathrm{Proj}_C\, \mathcal{P}_s$ as the non-commutative projective bundle corresponding to $s \in \mathcal{S}$, with fibration $\pi_s : \mathbf{P}_s \to C$. Setting $z = 0$, we have its divisor at infinity

$$i_s : D_s \hookrightarrow \mathbf{P}_s.$$

The divisor D_s has the structure of an ordinary (commutative) $\mathbb{P}^1$-bundle

$$\pi_s|_{D_s} = \tau_s \colon D_s \to C$$

equipped with a Γ-action on the fibres. Its complement $\mathbf{A}_s = \mathbf{P}_s \setminus D_s = \operatorname{Spec}_C \mathcal{A}_s$ is a non-commutative affine bundle.

The sheaf $\mathcal{P}_s$ is a sheaf of regular graded algebras in the sense of [1]; sheaf theory on $\mathbf{P}_s$ works in complete analogy with the absolute case discussed in [3]. The category of coherent sheaves $\operatorname{Coh}(\mathbf{P}_s)$ is by definition the quotient of the category of sheaves of finitely generated graded right $\mathcal{P}_s$-modules by the subcategory of sheaves of torsion $\mathcal{P}_s$-modules; we will sometimes refer to objects in this category as $\mathcal{P}_s$-modules. The trivial module, graded in degree n, defines the object $\mathcal{O}_{\mathbf{P}_s}(n) \in \operatorname{Coh}(\mathbf{P}_s)$; given a sheaf $\mathcal{E}$, its twists $\mathcal{E}(n)$ are obtained by shifting the grading. We have Ext groups as the derived functors of Hom, and also functors $\mathcal{E}xt^i(-, \mathcal{O}_{\mathbf{P}_s})$; the latter take values in the category of left $\mathcal{P}_s$-modules (compare [3]).

Pushforward

$$\pi_{s*} \colon \operatorname{Coh}(\mathbf{P}_s) \to \operatorname{Coh}^\Gamma(C)$$

along the morphism $\pi_s \colon \mathbf{P}_s \to C$ is defined in the usual way, as the coherent Γ-sheaf on C defined by sections over preimages of open sets of C, the section spaces being (right) $\mathbb{C}\Gamma$-modules; the action of Γ on C is taken to be trivial. The higher pushforwards $\mathrm{R}^p\pi_{s*}(-)$ are the derived functors of π_{s*}. Given a $\mathbf{P}_s$-module $\mathcal{E}$, we will also use the relative Hom-functor

$$\operatorname{Hom}_C(\mathcal{E}, -) \colon \operatorname{Coh}(\mathbf{P}_s) \to \operatorname{Coh}^\Gamma(C),$$

defined by homomorphisms on preimages of open sets in C, as well as its derived functors

$$\operatorname{Ext}^i_C(\mathcal{E}, -) \colon \operatorname{Coh}(\mathbf{P}_s) \to \operatorname{Coh}^\Gamma(C).$$

We also have a pullback functor

$$\pi_s^* \colon \operatorname{Coh}^\Gamma(C) \to \operatorname{Coh}(\mathbf{P}_s)$$

taking a sheaf of (right) $\mathbb{C}\Gamma$-modules $\mathcal{F}$ to the sheaf $\mathcal{F} \otimes_{\mathbb{C}\Gamma} \mathcal{P}_s$ of (right) $\mathcal{P}_s$-modules. The pair (π_s^*, π_{s*}) forms an adjoint pair as in the commutative case. Similarly, for the inclusion $i_s \colon D_s \to \mathbf{P}_s$, we have a pullback (restriction) functor

$$i_s^* \colon \operatorname{Coh}(\mathbf{P}_s) \to \operatorname{Coh}^\Gamma(D_s),$$

defined by factoring modules of local sections by the ideal $\langle z \rangle$ (recall that z is central), as well as a pushforward

$$i_{s*} \colon \operatorname{Coh}^\Gamma(D_s) \to \operatorname{Coh}(\mathbf{P}_s),$$

with z acting on local sections by zero. There is also a restriction functor to the finite part $\mathbf{A}_s$, defined by factoring the ideal $\langle z - 1 \rangle$.

Definition 2.4. *A π_s-free sheaf on $\mathbf{P}_s$ is an object $\mathcal{E} \in \mathrm{Coh}(\mathbf{P}_s)$, which admits an embedding*

$$\mathcal{E} \hookrightarrow \pi_s^*(\mathcal{U})(n)$$

for some $\mathcal{U} \in \mathrm{Coh}^\Gamma(C)$ and $n \in \mathbb{Z}$. A framed π_s-free sheaf $(\mathcal{E}, \varphi)$ on $(\mathbf{P}_s, D_s)$ is a π_s-free sheaf $\mathcal{E}$ on $\mathbf{P}_s$, together with a fixed isomorphism

$$\varphi\colon i_s^*\mathcal{E} \xrightarrow{\sim} \tau_s^*\mathcal{W},$$

on the divisor D_s at infinity, for some $\mathcal{W} \in \mathrm{Coh}^\Gamma(C)$.

Remark 2.5. If $\pi : \mathbf{P} \to \{*\}$ is a (non-commutative) projective space over a point, the π-free sheaves are exactly the torsion free ones (compare [3, Sect. 2]). To see this, note that a π-free sheaf is certainly torsion free, since it embeds into a locally free sheaf. Conversely, a torsion free sheaf embeds into some locally free sheaf, which in turn embeds into some $\mathcal{O}_{\mathbf{P}}^m(n)$.

Lemma 2.6. *If $\mathcal{E}$ is π_s-free, then $L^j i_s^*\mathcal{E} = 0$ for $j > 0$.*

Proof. As in the commutative case, the structure sheaf $i_{s*}\mathcal{O}_{D_s}$ has a resolution

$$0 \to \mathcal{O}_{\mathbf{P}_s}(-1) \xrightarrow{z} \mathcal{O}_{\mathbf{P}_s} \to i_{s*}\mathcal{O}_{D_s} \to 0,$$

which implies that $L^j i_s^*\mathcal{E} = 0$ for $j > 1$ for any $\mathcal{E} \in \mathrm{Coh}(\mathbf{P}_s)$, and also that $L^1 i_s^*$ is left exact. If $\mathcal{E}$ is π_s-free, applying the latter to an embedding $\mathcal{E} \hookrightarrow \pi_s^*(\mathcal{U})(n)$ gives the vanishing of L^1 also. □

3. Threefolds: The Results

3.1. Twisted quiver representations and quiver sheaves. Recall that, given a quiver with arrows $a \to b$ marked by objects $O_{ab} \in \mathcal{C}$ of an abelian tensor category $\mathcal{C}$, a representation of the marked quiver in $\mathcal{C}$ consists of a set of objects O_a of $\mathcal{C}$ associated to nodes, and a set of morphisms $\varphi_{ab} \in \mathrm{Hom}_{\mathcal{C}}(O_a \otimes O_{ab}, O_b)$ associated to the arrows $a \to b$. Representations of a marked quiver in the category $\mathrm{Coh}(X)$ of an algebraic variety X are also called *quiver sheaves* [12] on X.

In the specific context of classifying holomorphic D-branes on the threefold X and its deformations, the following quiver marked in $\mathrm{Coh}(C)$ will arise naturally. The quiver is the standard extended McKay quiver of [16], obtained from the original one by adding an extra leaf at each node with arrows in both directions. Using the data of the vector bundle $\mathcal{Q}$ on C, we mark this quiver in $\mathrm{Coh}(C)$ as follows:

- The marked A_n quiver for $n > 1$ is illustrated on Fig. 3.1; recall that in this case, there is a decomposition $\mathcal{Q} = \mathcal{Q}_1 \oplus \mathcal{Q}_2$ into a sum of line bundles, since the structure group of $\mathcal{Q}$ reduces to the diagonal torus.
- The marked A_1 quiver consists of only two nodes 0 and 1 and two arrows $0 \to 1$, $1 \to 0$ marked by the rank-two bundle $\mathcal{Q}^\vee$, as well as leaves marked as in the higher A_n case.
- For types D and E, arrows between nodes are all marked by the line bundle $\mathcal{Q}_0^\vee$, where $\mathcal{Q} = \mathcal{Q}_0^{\oplus 2}$; leaves are marked as before.

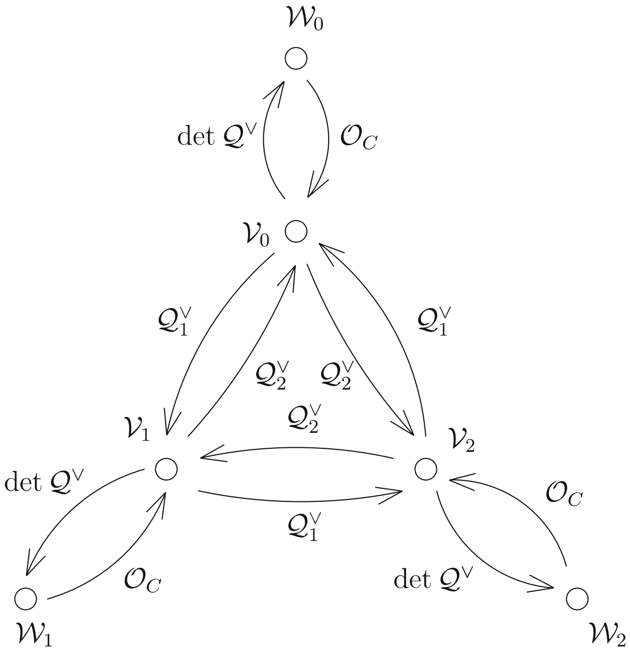

Fig. 3.1. The marked extended McKay quiver for A_2

3.2. The main classification result.

Theorem 3.1. *Given $s \in H^0(C, \mathcal{H})$, there is a 1-to-1 correspondence between the following sets of data:*

(1) *Isomorphism classes of framed π_s-free sheaves $(\mathcal{E}, \varphi)$ on $(\mathbf{P}_s, D_s)$.*
(2) *Quintuples $(\mathcal{V}, \mathcal{W}, \mathcal{B}, \mathcal{I}, \mathcal{J})$, where $\mathcal{W}, \mathcal{V}$ are coherent Γ-sheaves on C, and*

$$\mathcal{B} \in \mathrm{Hom}_C^\Gamma(\mathcal{V} \otimes \mathcal{Q}^\vee, \mathcal{V}),$$

$$\mathcal{I} \in \mathrm{Hom}_C^\Gamma(\mathcal{W}, \mathcal{V}),$$

$$\mathcal{J} \in \mathrm{Hom}_C^\Gamma(\mathcal{V} \otimes \det \mathcal{Q}^\vee, \mathcal{W}),$$

satisfying the following two conditions:
(a) *the ADHM relation*

$$\mathcal{B} \wedge \mathcal{B} + \mathcal{I} \circ \mathcal{J} + s = 0 \in \mathrm{Hom}_C^\Gamma(\mathcal{V} \otimes \det \mathcal{Q}^\vee, \mathcal{V}),$$

where

$$H^0(C, Z(\mathbb{C}\Gamma) \otimes \det \mathcal{Q}) \hookrightarrow \mathrm{Hom}_C^\Gamma(\mathcal{V} \otimes \det \mathcal{Q}^\vee, \mathcal{V})$$

is the natural embedding as the central subspace;
(b) *non-degeneracy: if $\mathcal{V}' \subset \mathcal{V}$ is a Γ-subsheaf such that $\mathcal{B}(\mathcal{V}' \otimes \mathcal{Q}^\vee) \subset \mathcal{V}'$ and $\mathcal{I}\mathcal{W} \subset \mathcal{V}'$, then $\mathcal{V}' = \mathcal{V}$.*

Sets of quintuples are identified under the action of invertible elements of $\mathrm{Hom}_C^\Gamma(\mathcal{V}, \mathcal{V})$.

(3) *Representations* $(\{\mathcal{V}_a\}, \{\mathcal{W}_a\}, \{\mathcal{B}_{ab}\}, \{\mathcal{I}_a\}, \{\mathcal{J}_a\})$ *in* $\mathrm{Coh}(C)$ *of the marked McKay-type quiver introduced in 3.1, satisfying*
   (a) *the ADHM relations*

$$\sum_b \epsilon_{ab}\mathcal{B}_{ba} \circ \mathcal{B}_{ab} + \mathcal{I}_a \circ \mathcal{J}_a + s_a = 0 \in \mathrm{Hom}_C(\mathcal{V}_a \otimes \det \mathcal{Q}^\vee, \mathcal{V}_a)$$

   at each node a, where $\epsilon_{ab} \in \{\pm 1\}$ *is a standard assignment of signs to arrows with* $\epsilon_{ab} = -\epsilon_{ba}$*, and* $s_a = P_{\eta_a}(s)$ *is the projected section corresponding to the simple root* η_a*, and*
   (b) *non-degeneracy: if* $\{\mathcal{V}'_a\}$ *is a* $\mathcal{B}$*-invariant set of subsheaves containing the images of* $\mathcal{I}_a$*'s, then* $\mathcal{V}'_a = \mathcal{V}_a$ *at all nodes.*
   Two representations are identified under invertible elements of $\prod_a \mathrm{Hom}_C(\mathcal{V}_a, \mathcal{V}_a)$.

If $s \in \mathcal{S} = H^0(C, \mathcal{H}_0)$ *is a deformation parameter of the threefold* $X = X_0$*, then the same data also parametrizes*

(4) *Certain objects in* $\mathcal{D}(\mathrm{Coh}\, X_s)$*, the derived category of coherent sheaves on* X_s*.*

Proof. The equivalence (1) $\iff$ (2) follows from a version of the relative Beilinson resolution for the non-commutative projective bundle $\mathbf{P}_s \to C$; details are given in Sect. 4.1. McKay's definition of the quiver describing the representation theory of Γ implies (2) $\iff$ (3) in the standard way. Finally the mapping (1) $\implies$ (4) in the geometric case $s \in \mathcal{S} = H^0(C, \mathcal{H}_0)$ is given by a derived equivalence to be discussed in Sect. 4.2. $\square$

Remark 3.2. As $X = X_0$ and its deformations X_s for $s \in \mathcal{S}$ are not projective, one needs to rigidify before holomorphic D-branes, in other words objects in $\mathcal{D}^b(X_s)$ have a sensible moduli space. For the central fibre $X = X_0$, a crepant resolution of the singular threefold $\mathcal{Q}/\Gamma$, one has a derived equivalence [5]

$$\mathcal{D}(X_0) \cong \mathcal{D}^\Gamma(\mathcal{Q})$$

between the derived categories of coherent sheaves on X_0 and that of Γ-equivariant sheaves on the total space of the bundle $\mathcal{Q} \to C$. One can easily rigidify on the latter by considering Γ-sheaves on the projective bundle $\mathbf{P}_0 = \mathbb{P}(\mathcal{Q} \oplus \mathcal{O}_C) \to C$, framed on the divisor at infinity $D_0 = \mathbb{P}(\mathcal{Q}) \hookrightarrow \mathbf{P}_0$. Theorem 3.1 is the appropriate generalization of this approach which also works for deformations: for the analogous problem on X_s, we consider framed sheaves on the non-commutative projective bundle $\mathbf{P}_s \to C$.

In the surface case, this approach was used earlier in [3]. To quote the result, let $\lambda \in Z(\mathbb{C}\Gamma)$. Then for Γ-modules V, W, Nakajima's non-singular quiver variety $\mathcal{M}_{V,W,\lambda}$ parametrizes torsion free sheaves on the non-commutative space $\mathbb{P}^2_\lambda = \mathrm{Proj}\ P_\lambda$, framed on the commutative Γ-line at ∞. This statement generalizes earlier work of [10,15,17,18,13] and others. The origin of all such results is of course the ADHM classification [2] of finite-action $\mathrm{SU}(\dim(W))$-instantons on $\mathbb{R}^4$ of charge $\dim(V)$.

3.3. Some holomorphic D-branes on ADE fibrations over $\mathbb{A}^1$. If $C \cong \mathbb{A}^1$, Theorem 3.1 can in some cases be re-written in terms of classical quiver representations: representations of a quiver in vector spaces. This will give an interpretation of an assertion of [6,14,21].

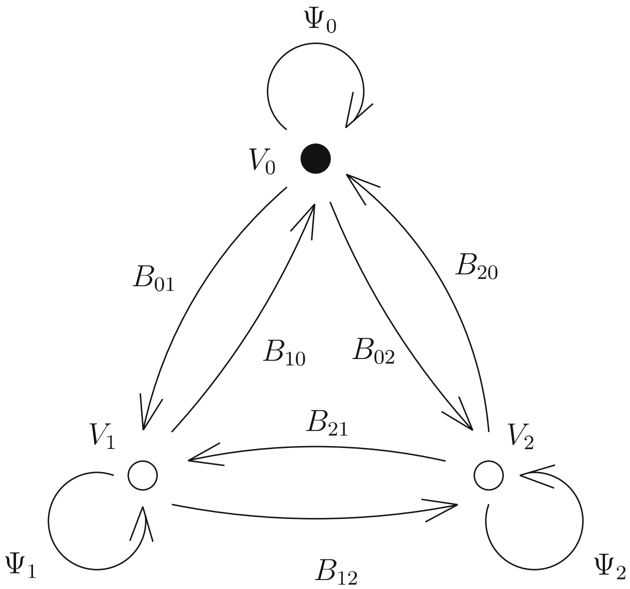

Fig. 3.2. A representation of the affine $N = 1$ A_2 quiver

Recall that for $C \cong \mathbb{A}^1$, a deformation parameter $s \in \mathcal{S}$ of the central fibre $X_0 = \mathbb{A}^1 \times Y$ can be specified by a set of polynomials $\{\Theta_a \in \mathbb{C}[t] : a \in \Delta\}$ indexed by the vertices of the affine quiver, subject to $\sum_a \delta_a \cdot \Theta_a = 0$. The following definition is due to Cachazo–Katz–Vafa [6,14].

Definition 3.3. *The affine $N = 1$ ADE quiver is the McKay quiver extended by a loop $a \to a$ at each vertex. For a (finite-dimensional) representation $(\{V_a\}, \{B_{ab}\}, \{\Psi_a\})$ of this quiver, the ADHM-type relations are*

$$\sum_b \epsilon_{ab} B_{ba} B_{ab} + \Theta_a(\Psi_a) = 0 \in \mathrm{Hom}(V_a, V_a) \tag{3.1}$$

at each vertex $a \in \Delta$ of the quiver, where $\Theta_a(\Psi_a)$ is to be interpreted as the evaluation of a polynomial on an endomorphism of V_a, as well as

$$\Psi_a B_{ba} = B_{ba} \Psi_b \in \mathrm{Hom}(V_a, V_b) \tag{3.2}$$

along each arrow $a \to b$ of the quiver Δ.

Consider quadruples $(\{V_a\}, \{B_{ab}\}, \{\Psi_a\}, \mathbf{v_0})$, where $(\{V_a\}, \{B_{ab}\}, \{\Psi_a\})$ is a representation of the affine $N = 1$ ADE quiver satisfying the ADHM-type relations, and $\mathbf{v_0} \in V_0$ is a fixed vector in the vector space attached to the affine node. Call a quadruple non-degenerate if there is no (B, Ψ)-invariant collection of subspaces $\{V_a' \subset V_a\}$ with $\mathbf{v_0} \in V_0'$.

Theorem 3.4. *Equivalence classes of non-degenerate quadruples $(\{V_a\}, \{B_{ab}\}, \{\Psi_a\}, \mathbf{v_0})$ satisfying the ADHM relations, identified under the action of $\prod_a \mathrm{GL}(V_a)$, parametrize certain objects in $\mathcal{D}(\mathrm{Coh}\ X_s)$, holomorphic D-branes on the threefold X_s.*

Proof. Quiver sheaf data on C parametrize certain branes on X_s by Theorem 3.1. The correspondence between representations of the $N=1$ ADE quiver and a special class of quiver sheaf data will be discussed in Sect. 4.3. □

Remark 3.5. As explained in [6], the quiver relations (3.1)-(3.2) come from the natural superpotential of the quiver gauge theory on Δ, involving adjoint fields Ψ_a as well as bifundamental fields B_{ab}.

Remark 3.6. Let the finite $N=1$ ADE quiver be obtained from the affine one by deleting the affine node. Representations of the finite $N=1$ ADE quiver, satisfying the ADHM-type relations (3.1)-(3.2), parametrize holomorphic D-branes supported on exceptional fibres of $f_s\colon X_s\to\bar{X}_s$. This follows from the statement that the vanishing of the affine component of $\mathcal{V}$ forces all other $\mathcal{V}_a$ to be supported on points $P\in C$ at which some projected section $p_\eta(s)$ vanishes for some positive root $\eta\in R_+$, in other words on points of the base curve over which the surface fiber $\pi_s^{-1}(P)$ contains exceptional curves. Observing that the section $s\in H^0(C,\mathbb{Z}(\mathbb{C}\Gamma)\otimes\det\mathcal{Q})$ is central in $\mathrm{Hom}_C^\Gamma(\mathcal{V}\otimes\det\mathcal{Q}^\vee,\mathcal{V})$, so commutes with all components of $\mathcal{B}$, the latter statement is essentially proved in [6, 4.1–4.2]. This establishes a direct link to [14,21], according to which (in the generic case) irreducible representations of the finite $N=1$ quiver with the given relations parametrize exceptional components of the contraction $f_s\colon X_s\to\bar{X}_s$.

4. Proofs

4.1. The Beilinson argument. The aim of this section is to prove of the equivalence (1) $\iff$ (2) of the classification result Theorem 3.1 via an analysis of framed π_s-free sheaves on $\mathbf{P}_s$.

Given $s\in H^0(C,\mathcal{H})$, recall the sheaf of algebras $\mathcal{P}_s$ on the curve C, and the associated non-commutative bundle $\pi_s\colon\mathbf{P}_s\to C$. Define $\mathcal{P}_s$-modules $\mathcal{T}_i$ by

$$\mathcal{T}_0=\mathcal{O}_{\mathbf{P}_s},$$

$$0\longrightarrow\mathcal{O}_{\mathbf{P}_s}\longrightarrow\pi_s^*(\mathcal{Q}\oplus\mathcal{O}_C)(1)\longrightarrow\mathcal{T}_1\longrightarrow 0,\tag{4.3}$$

$$\mathcal{T}_2=\pi_s^*(\det\mathcal{Q})(3).$$

Proposition 4.1. *A π_s-free sheaf $\mathcal{E}$ on $\mathbf{P}_s$, framed on the divisor D_s, is the cohomology of a monad*

$$\pi_s^*\,\mathrm{Ext}^1_C\,(\mathcal{T}_2(-1),\mathcal{E})\,(-1)\to\pi_s^*\,\mathrm{Ext}^1_C\,(\mathcal{T}_1,\mathcal{E})\to\pi_s^*\,\mathrm{Ext}^1_C\,(\mathcal{T}_0(1),\mathcal{E})\,(1)$$

of $\mathcal{P}_s$-modules.

Proof. Given a $\mathcal{P}_s$-module $\mathcal{F}$, a Koszul duality argument, in an analogous way to the absolute case in [3, Sect. 7] following [4, Thm. 2.6.1], leads to a Beilinson-type spectral sequence with E_1 term

$$E_1^{p,q}=\pi_s^*\,\mathrm{Ext}^q_C\left(\mathcal{T}_{-p}(p),\mathcal{F}\right)(p),$$

nonzero only for $-2\le p\le 0,\ 0\le q\le 2$, converging to $\mathcal{F}$ in the limit. The vanishing results

$$\mathrm{Ext}^q_C\left(\mathcal{T}_{-p}(p),\mathcal{E}(-1)\right)=0\quad\text{for } q=0,2,\ p=-1,-2$$

which follow from the existence of the framing of $\mathcal{E}$ on the divisor D_s (compare [13, Lemma 6.2], [3, Lemma 4.2.12]), reduce the spectral sequence for $\mathcal{F} = \mathcal{E}(-1)$ to the monad given in the statement. Details are left to the reader. □

We also record an auxiliary result.

Lemma 4.2. *There are natural isomorphisms*

$$\mathrm{Hom}_{\mathbf{P}_s}\big(\pi_s^* \det \mathcal{Q}^\vee, \pi_s^* \mathcal{Q}^\vee(1)\big) \cong \mathrm{Hom}_{\mathbf{P}_s}\big(\pi_s^* \mathcal{Q}^\vee, \mathcal{O}_{\mathbf{P}_s}(1)\big) \cong \mathrm{Hom}_C^\Gamma\big(\mathcal{Q}^\vee, \mathcal{Q}^\vee \oplus \mathcal{O}_C\big).$$

Proof. The first isomorphism follows from Lemma 4.3 below. The second one follows from adjunction for the pair (π_s^*, π_{s*}), together with

$$\pi_{s*}\mathcal{O}_{\mathbf{P}_s}(1) \cong \mathcal{P}_{s,1} \cong (\mathcal{Q}^\vee \oplus \mathcal{O}_C) * \mathbb{C}\Gamma \in \mathrm{Coh}^\Gamma(C),$$

an identity well known from the commutative context. □

Lemma 4.3. *Let $\mathcal{Q}$ be a rank-two bundle on a (commutative) space. Then there is a natural isomorphism*

$$\mathcal{Q} \otimes \det \mathcal{Q}^\vee \cong \mathcal{Q}^\vee.$$

Proof. The embedding $\iota\colon \det \mathcal{Q}^\vee \to \mathcal{Q}^\vee \otimes \mathcal{Q}^\vee$ induces a natural map

$$\mathrm{Hom}\big(\det \mathcal{Q}^\vee, \det \mathcal{Q}^\vee\big) \to \mathrm{Hom}\Big(\det \mathcal{Q}^\vee, (\mathcal{Q}^\vee)^{\otimes 2}\Big) \cong \mathrm{Hom}\big(\mathcal{Q} \otimes \det \mathcal{Q}^\vee, \mathcal{Q}^\vee\big).$$

The image of the identity of the first Hom-group gives a natural morphism as in the statement, which can be checked on a local basis to be an isomorphism. □

Now return to the context of the classification result Theorem 3.1, and consider a quintuple $(\mathcal{V}, \mathcal{W}, \mathcal{B}, \mathcal{I}, \mathcal{J})$ as in Theorem 3.1(2); recall that

$$\mathcal{W}, \mathcal{V} \in \mathrm{Coh}^\Gamma(C),$$

and

$$\mathcal{B} \in \mathrm{Hom}_C^\Gamma(\mathcal{V} \otimes \mathcal{Q}^\vee, \mathcal{V}),$$

$$\mathcal{I} \in \mathrm{Hom}_C^\Gamma(\mathcal{W}, \mathcal{V}),$$

$$\mathcal{J} \in \mathrm{Hom}_C^\Gamma(\mathcal{V} \otimes \det \mathcal{Q}^\vee, \mathcal{W}).$$

Let

$$c \in \mathrm{Hom}_{\mathbf{P}_s}\big(\pi_s^* \det \mathcal{Q}^\vee, \pi_s^* \mathcal{Q}^\vee(1)\big), \quad d \in \mathrm{Hom}_{\mathbf{P}_s}\big(\pi_s^* \mathcal{Q}^\vee, \mathcal{O}_{\mathbf{P}_s}(1)\big)$$

denote the images, under the isomorphisms of Lemma 4.2, of the canonical element

$$\mathrm{Id} \in \mathrm{Hom}_C^\Gamma(\mathcal{Q}^\vee, \mathcal{Q}^\vee) \subset \mathrm{Hom}_C^\Gamma\big(\mathcal{Q}^\vee, \mathcal{Q}^\vee \oplus \mathcal{O}_C\big).$$

Note also that we have a fixed section

$$z \in \mathrm{Hom}_{\mathbf{P}_s}(\mathcal{O}_{\mathbf{P}_s}, \mathcal{O}_{\mathbf{P}_s}(1)).$$

Define

$$a = \begin{pmatrix} \pi_s^* \left(\mathcal{B} \circ (\mathrm{Id}_{\mathcal{V}} \otimes \iota)\right) \otimes z - \pi_s^*(\mathrm{Id}_{\mathcal{V}}) \otimes c\,(-1) \\ \pi_s^*(\mathcal{J}) \otimes z \end{pmatrix} : \pi_s^*(\mathcal{V} \otimes \det \mathcal{Q}^\vee)(-1)$$
$$\longrightarrow \pi_s^*(\mathcal{V} \otimes \mathcal{Q}^\vee \oplus \mathcal{W}),$$

where $\iota\colon \det \mathcal{Q}^\vee \to (\mathcal{Q}^\vee)^{\otimes 2}$ is the natural map. Define similarly

$$b = \left(\pi_s^*(\mathcal{B}) \otimes z + \pi_s^*(\mathrm{Id}_{\mathcal{V}}) \otimes d \;\; \pi_s^*(\mathcal{I}) \otimes z\right) : \pi_s^*(\mathcal{V} \otimes \mathcal{Q}^\vee \oplus \mathcal{W}) \to \pi_s^*(\mathcal{V})(1),$$

to obtain the chain of morphisms

$$\pi_s^*(\mathcal{V} \otimes \det \mathcal{Q}^\vee)(-1) \xrightarrow{a} \pi_s^*(\mathcal{V} \otimes \mathcal{Q}^\vee \oplus \mathcal{W}) \xrightarrow{b} \pi_s^*(\mathcal{V})(1). \tag{4.4}$$

The following result completes the proof of the equivalence (1) $\iff$ (2) of the classification result Theorem 3.1.

Proposition 4.4. *If the quintuple satisfies the ADHM relation, then* (4.4) *is a complex of $\mathcal{P}_s$-modules. Furthermore, it is a monad defining a framed π_s-free sheaf $\mathcal{E}$ if and only if the quintuple $(\mathcal{V}, \mathcal{W}, \mathcal{B}, \mathcal{I}, \mathcal{J})$ is non-degenerate. Conversely, every π_s-free $\mathcal{P}_s$-module $\mathcal{E}$, framed on D_s, arises from this construction.*

Proof. The standard direct computation shows that $b \circ a = 0$ is equivalent to the ADHM relation. The proof of the equivalence of the monad property and non-degeneracy is analogous to the absolute case [3, Sect. 4.1]. For the converse, given a framed sheaf $(\mathcal{E}, \varphi)$, let $\mathcal{V} = \mathrm{Ext}^1_C(\mathcal{O}_{\mathbf{P}_s}(1), \mathcal{E})$. Then by Proposition 4.1, $\mathcal{E}$ is the middle cohomology of the monad

$$\pi_s^*(\mathcal{V} \otimes \det \mathcal{Q}^\vee)(-1) \to \pi_s^* \,\mathrm{Ext}^1_C\,(\mathcal{T}_1, \mathcal{E}) \to \pi_s^*\mathcal{V}(1).$$

The usual arguments [13, Theorem 6.7] show that, since $\mathcal{E}$ is framed on D_s, this monad is isomorphic to a monad of the form (4.4) for some quintuple $(\mathcal{V}, \mathcal{W}, \mathcal{B}, \mathcal{I}, \mathcal{J})$. □

4.2. A derived equivalence. In this section we complete the proof of Theorem 3.1 by establishing the missing link (1) $\implies$ (4).

Proposition 4.5. *Let $s \in \mathcal{S}$ be a deformation parameter of the central fibre $X = X_0$. There is a distinguished equivalence of triangulated categories*

$$\mathcal{D}(\mathrm{Coh}\; X_s) \cong \mathcal{D}(\mathrm{Mod}\; \mathcal{A}_s),$$

where Mod $\mathcal{A}_s$ *is the category of sheaves of finitely generated right $\mathcal{A}_s$-modules, and* $\mathcal{D}(-)$ *denotes the bounded derived category on both sides.*

Proof. This assertion is a fibered version of the analogous two-dimensional equivalence proved in [11], and the proof carries over verbatim. A deformation argument starting from the central fibre $X = X_0$ shows that a certain specific component M_s of a fine moduli space of torsion sheaves on $\mathbf{A}_s$ maps by a semi-small birational map to the singular variety $\bar{X}_s$. By [20], generalizing an argument of [5], this implies that M_s is a crepant resolution of $\bar{X}_s$, and one has a derived equivalence

$$\mathcal{D}(\mathrm{Coh}\; M_s) \cong \mathcal{D}(\mathrm{Mod}\; \mathcal{A}_s)$$

defined by the universal sheaf. But since X_s is the unique crepant resolution of $\bar{X}_s$, necessarily $M_s \cong X_s$ and the proposition follows. Details are left to the reader. □

This equivalence gives the mapping (1) $\Longrightarrow$ (4) of Theorem 3.1 from framed π_s-free sheaves on $\mathbf{P}_s$ to objects in $\mathcal{D}(\mathrm{Coh}\ X_s)$. Indeed, a right $\mathcal{P}_s$-module can be restricted to the affine part $\mathbf{A}_s$ to give a right $\mathcal{A}_s$-module, and then mapped using the derived equivalence to an object in $\mathcal{D}(\mathrm{Coh}\ X_s)$, in other words a holomorphic D-brane on X_s.

4.3. Fibrations over the affine line. In this section, we take a fibration $X_s \to C \cong \mathbb{A}^1$ and discuss the proof of Theorem 3.4. From Theorem 3.1, we know that certain holomorphic D-branes on X_s are classified by non-degenerate quintuples $(\mathcal{V}, \mathcal{W}, \mathcal{B}, i, j)$ satisfying the ADHM equation. Consider the subclass of representations in $\mathrm{Coh}(\mathbb{A}^1)$ with the simplest possible framing $\mathcal{W} \cong \mathcal{O}_{\mathbb{A}^1}$ and $\mathcal{V}$ a torsion Γ-sheaf on $\mathbb{A}^1$. It follows that $\mathcal{J} = 0$ and $\mathcal{I} \in H^0(\mathbb{A}^1, \mathcal{V}^\Gamma)$. Decompose $\mathcal{V}$ and the map B into Γ-components to obtain torsion sheaves $\mathcal{V}_a$ and sheaf homomorphisms $\mathcal{B}_{ab} \colon \mathcal{V}_a \to \mathcal{V}_b$ indexed by nodes and edges of the McKay quiver.

Set $V_a = H^0(\mathbb{A}^1, \mathcal{V}_a)$, and let $B_{ab} = H^0(\mathcal{B}_{ab}) \colon V_a \to V_b$ be the map on global sections induced by $\mathcal{B}_{ab}$. Let $\mathbf{v_0} \in V_0$ be the section corresponding to $\mathcal{I}$. Let also $\Psi_a \colon V_a \to V_a$ be the map induced by multiplication by the section $t \in H^0(\mathbb{A}^1, \mathcal{O}_{\mathbf{A}_1}) \cong \mathbb{C}[t]$. Theorem 3.4 follows from Theorem 3.1, together with

Proposition 4.6. *The map*

$$(\mathcal{V}, \mathcal{O}_C, \mathcal{B}, 0, 0) \mapsto (\{V_a\}, \{B_{ab}\}, \{\Psi_a\}, \mathbf{v_0} \in V_0)$$

sets up a one-to-one correspondence from this restricted set of quiver ADHM data to representations of the affine $N = 1$ ADE quiver satisfying the relations (3.1)-(3.2).

Proof. Given $(\mathcal{V}, \mathcal{B})$, the edge relations (3.2) $\Psi_a B_{ba} = B_{ba} \Psi_b$ for the data $(\{V_a\}, \{B_{ab}\}, \{\Psi_a\})$ hold by definition. Further, the ADHM equation for $(\mathcal{V}, \mathcal{B})$ is

$$\mathcal{B} \wedge \mathcal{B} + s = 0 \in \mathrm{Hom}(\mathcal{V}, \mathcal{V} \otimes \det \mathcal{Q}),$$

which in Γ-components says that

$$\sum_b \epsilon_{ab} \mathcal{B}_{ba} \circ \mathcal{B}_{ab} + s_a = 0 \in \mathrm{Hom}(\mathcal{V}_a, \mathcal{V}_a).$$

Replacing s_a by the polynomial Θ_a, and remembering that the effect of $t \in H^0(\mathcal{O}_{\mathbb{A}^1})$ on $H^0(\mathcal{V})$ is exactly Ψ_a, for global sections we obtain

$$\sum_b \epsilon_{ab} B_{ba} \circ B_{ab} + \Theta_a(\Psi_a) = 0 \in \mathrm{Hom}(V_a, V_a)$$

which is exactly relation (3.1) for the node a.

Conversely, given a representation $(\{V_a\}, \{B_{ab}\}, \{\Psi_a\}, \mathbf{v_0} \in V_0)$ of the $N = 1$ ADE quiver, define torsion sheaves attached to the nodes by

$$\mathcal{V}_a = \mathrm{coker}\left(V_a \otimes \mathcal{O}_{\mathbb{A}^1} \xrightarrow{1\otimes t - \Psi_a \otimes 1} V_a \otimes \mathcal{O}_{\mathbb{A}^1}\right).$$

Using Lemma 4.7 below, for adjacent nodes a, b we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_a \otimes \mathcal{O}_{\mathbb{A}^1} & \xrightarrow{1\otimes t - \Psi_a \otimes 1} & V_a \otimes \mathcal{O}_{\mathbb{A}^1} & \longrightarrow & \mathcal{V}_a & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle B_{ab}\otimes 1} & & \downarrow {\scriptstyle B_{ab}\otimes 1} & & & & \\ 0 & \longrightarrow & V_b \otimes \mathcal{O}_{\mathbb{A}^1} & \xrightarrow{1\otimes t - \Psi_b \otimes 1} & V_b \otimes \mathcal{O}_{\mathbb{A}^1} & \longrightarrow & \mathcal{V}_b & \longrightarrow & 0 \end{array}$$

which, by commutativity $\Psi_a B_{ba} = B_{ba}\Psi_b$, induces a map $\mathcal{B}_{ab}\colon \mathcal{V}_a \to \mathcal{V}_b$. The converse of the above argument shows that the ADHM relation follows from the relations (3.1). By Lemma 4.7, the two constructions are inverses to each other. □

The proof used the elementary

Lemma 4.7. *Given a torsion sheaf $\mathcal{V}$ on $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$, let $V = H^0(\mathbb{A}^1, \mathcal{V})$ and let $\Psi\colon V \to V$ be the map given by multiplication by $t \in H^0(\mathcal{O}_{\mathbb{A}^1})$. Then the sequence of sheaves*

$$0 \longrightarrow V \otimes \mathcal{O}_{\mathbb{A}^1} \xrightarrow{1\otimes t - \Psi\otimes 1} V \otimes \mathcal{O}_{\mathbb{A}^1} \xrightarrow{c} \mathcal{V} \longrightarrow 0$$

is exact on $\mathbb{A}^1$, where $c\colon H^0(\mathcal{V}) \otimes \mathcal{O}_{\mathbb{A}^1} \to \mathcal{V}$ is the canonical map. Conversely, given a vector space with an endomorphism (V, Ψ), the exact sequence defines a torsion sheaf $\mathcal{V}$ on $\mathbb{A}^1$, and the two constructions are mutual inverses.

Remark 4.8. In this lemma, $\mathcal{V} \cong \mathcal{O}_Z$ is a structure sheaf of a 0-dimensional subscheme $Z \subset \mathbb{A}^1$ if and only if Ψ is a regular endomorphism. Their moduli space is

$$\operatorname{Mat}(n, \mathbb{C})/\!/\operatorname{GL}(n, \mathbb{C}) \cong \{\text{regular endomorphisms}\}/\operatorname{GL}(n, \mathbb{C}) \cong \mathbb{A}^n \cong (\mathbb{A}^1)^{[n]},$$

where the map is given by taking the coefficients of the characteristic polynomial of Ψ, which is also the equation of the corresponding subscheme.

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