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Calabi–Yau threefolds with a curve of singularities and counterexamples to the Torelli problem II

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Introduction

This paper is a continuation of [**15**]. In that paper, I introduced a general framework which allows one to produce 'weak' counterexamples to Torelli for Calabi–Yau threefolds: deformation families containing non-isomorphic varieties Y_t, Y_t^+ with isomorphic Hodge theory on the third cohomology. The varieties arise as deformations of threefolds Y that are resolutions of singular varieties X with rather special properties (cf. Section 1). In [**15**], I discussed two families containing suitable X that do provide a counterexample and a third family with remarkably similar properties where however the existence of a nontrivial generic automorphism destroys the counterexample. This shows that explicit examples are necessary; there is a precise set of conditions one needs to check carefully in order to obtain counterexamples to Torelli (Theorem 1·1). In fact, there exist several families with renitent automorphisms (Remark 4·5).

The conditions involve two things: log-extremal contractions on the resolution Y and the automorphism group of X. Both contractions and automorphisms can be conveniently investigated if X is embedded in a suitable ambient space; it is natural to restrict to families arising as complete intersections of low codimension in weighted projective spaces.

The main result of this paper is that Calabi–Yau threefolds with the prescribed class of singularities embedded as low-codimensional complete intersections in

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weighted projective spaces can be classified (Theorem 1), using an explicit Riemann–Rock formula (Section 2) and the Hilbert series of a complete intersection variety (Section 3) following Reid–Fletcher. Contractions on the resolutions of these varieties can be studied using toric geometry; their automorphism groups can be determined by doing explicit algebra with polynomials. The final list of counterexamples to Torelli that one obtains this way is the following (for the proof, see Section 4):

Theorem 0·1. *Let X be a general member of one of the families*

$$\begin{array}{lll} X_8 \subset \mathbb{P}^4[1^2,2^3], & X_{4,6} \subset \mathbb{P}^5[1^2,2^4], & X_{4,4,4} \subset \mathbb{P}^6[1^2,2^5], \\ X_{12} \subset \mathbb{P}^4[1,2^2,3,4], & X_{6,6} \subset \mathbb{P}^5[1,2^4,3], & X_{4,4,6} \subset \mathbb{P}^6[1,2^5,3], \\ X_{12} \subset \mathbb{P}^4[1^2,2^2,6], & X_{6,8} \subset \mathbb{P}^5[1,2^3,3,4], & X_{4,6,6} \subset \mathbb{P}^6[2^5,3^2]. \end{array}$$

Let Y be the crepant resolution of X. Then the period map is a finite map of degree at least two from the set of deformations of Y to the period domain modulo monodromy.

The really important issue here is perhaps not the list itself, but rather the way in which it is obtained: the search based on the Hilbert series of graded rings, coupled to the power of toric geometry and explicit calculations. The author hopes that this method will prove to be useful in different contexts as well.

Notation and definitions

A *Calabi–Yau threefold* is a normal projective threefold X with canonical Gorenstein singularities, satisfying $K_X \sim 0$ and $H^1(X, \mathcal{O}_X) = 0$. These conditions imply in particular that $\mathrm{Pic}\,(X) \cong H^2(X,\mathbb{Z})$. The *nef cone* of X is the closed cone generated by ample classes in $\mathrm{Pic}\,(X) \otimes \mathbb{R} \cong H^2(X,\mathbb{R})$. Faces of this cone correspond in most cases to contractions on X, for a classification and the Type convention see [**17**].

The *weighted projective space* $\mathbb{P} = \mathbb{P}^m[w_0,\ldots,w_m]$ is the quotient of $\mathbb{C}^{n+1}\backslash\{0\}$ by the $\mathbb{C}^*$-action having the given positive integer weights $\{w_i\}$. It is a normal projective variety with quotient singularities. It is called *well-formed* if the greatest common divisor of every set of n weights is one. A *weighted complete intersection* $X = X_{d_1,\ldots,d_c} \subset \mathbb{P}^m[w_0,\ldots,w_m]$ is defined by the vanishing of a regular sequence of homogeneous polynomials $f_1,\ldots,f_c$ of the appropriate (weighted) degrees. X is called *quasi-smooth* if the affine cone $CX \subset \mathbb{C}^{n+1}$ over X is smooth apart from its vertex. In this case, X is also a normal projective variety with only quotient singularities. X is called *well-formed* if the ambient space $\mathbb{P}$ is well-formed and X does not contain a $c+1$-codimensional singular stratum of P, where c is the codimension of X in $\mathbb{P}$. If X is well-formed and quasi-smooth then the adjunction formula holds in the form

$$K_X \sim \mathcal{O}_X\left(\sum d_i - \sum w_j\right).$$

For more details on this material, consult Fletcher [**6**].

Finally, $H^*(X)$ and $H_*(X)$ denote integer (co)homology modulo torsion.

1. *The framework*

Assume that Y is a smooth Calabi–Yau threefold containing a surface E ruled over a curve C of genus g. Assume that E can be contracted inside Y by a log-extremal contraction given by the divisor $H \in \operatorname{Pic}(Y)$:

$$\begin{array}{cccc} \varphi_{|H|}: & Y & \to & X \\ & \cup & & \cup \\ & E & \to & C. \end{array}$$

By [**17**, **18**], C is smooth and the singularities of X along C are generically cA_1 or cA_2, the latter only occuring under special circumstances.

Let $\mathscr{Y} \to S$ be the the Kuranishi family of Y; by Unobstructedness, the base S is smooth. Let S_E denote the subspace of S that corresponds to deformation directions along which E deforms together with the deformation.

In [**15**], I proved the following theorem.

Theorem 1·1. *Assume that the following conditions are satisfied.*

(1) *The genus g of C is at least* 2.

(2) *Every automorphism of the linear space* $\operatorname{Pic}_{\mathbb{R}}(Y)$, *fixing the nef cone of Y and mapping faces of this cone to other faces of the same type, must be the trivial one.*

(3) *X has no involutions i with the following property: if j denotes the corresponding involution on Y, then the fixed locus of the induced map $j^*: S \to S$ equals S_E.*

Then (*weak*) *global Torelli fails for Y: the period map from deformations of Y modulo isomorphisms to the period domain is of degree at least two onto its image in the appropriate period domain.*

Sketch Proof. The first condition ensures that for a general one-parameter deformation $\mathscr{Y}$ of Y, a finite positive number of fibres of the ruling $E \to C$ deform. These rational curves can be flopped in the family to obtain a new deformation $\mathscr{Y}^+$ of Y [**15**, 1·3]. Fibres Y_t , Y_t^+ of the two families have isomorphic Hodge structures [**10**, 4·12–13]. Using the other two assumptions, Y_t and $Y_t^+$are not isomorphic for general t by [**15**, 4·2], thus proving the conclusion.

Condition (1) is easy to check in a concrete situation. However, conditions (2) and (3) are in general harder. They can be conveniently investigated for hypersurfaces or more generally complete intersections in projective or weighted projective spaces. On one hand, this condition ensures that the Picard number of Y is two, which makes condition (2) easy to deal with (cf. Proposition 4·2). On the other hand, automorphism groups of complete intersections can be computed explicitly (Proposition 4·4).

2. *A Riemann–Roch formula*

From now on, I make the assumption that the singular Calabi–Yau variety X is a quasi-smooth well-formed weighted complete intersection

$$X_{d_1,\ldots,d_c} \subset \mathbb{P}^m[w_0, \ldots ,w_m]$$

of codimension at most *four* and that the curve C is a curve of cA_1-singularities of X. The latter is equivalent to E being geometrically ruled over C. It follows that the Picard number of X is one and that X is not factorial but only $\mathbb{Q}$-factorial. Let $D \in \operatorname{Weil}(X) \setminus \operatorname{Pic}(X)$ denote the ample $\mathbb{Q}$-Cartier class (unique up to torsion) which is Cartier away from the singular curve and non-divisible as a Weil divisor. The aim of this Section is to prove the following Riemann–Roch type expression for $h^0(X, nD)$, the number of sections of the divisorial sheaf $\mathscr{O}_X(nD)$ (cf. [**12**]).

PROPOSITION 2·1. *The number of sections of the divisorial sheaf $\mathcal{O}_X(nD)$ is given by the following formula*:

$$h^0(X, nD) = \tfrac{1}{6}n^3D^3 + \tfrac{1}{12}nc_2(X)\cdot D - \tfrac{1}{8}nkI \,(n \equiv 1 \bmod 2),$$

where I is an indicator function and, as usual, $c_2(X)\cdot D$ is to be calculated on any resolution (*e.g. on Y*).

Proof. As in the previous section, let $f\colon Y \to X$ be the blowup of C on X. Note that $H_2(Y)$ and $H^2(Y) \cong \operatorname{Pic}(Y)/\text{torsion}$ are free rank-two dual $\mathbb{Z}$-modules. Let $H = f^*(2D)$, a primitive (non-divisible) class in $H^2(Y)$; let l be the class of the fibre of the ruling of E.

LEMMA 2·2. *l is a non-divisible class in $H_2(Y)$, whereas the class $H+E$ is 2-divisible in $H^2(Y)$.*

Proof. Assume that $l = 2f$ for some $f \in H_2(Y)$, then the intersection numbers are

$$H\cdot f = 0,\; E\cdot f = -1.$$

Hence f is necessarily primitive, so one can choose a class $r \in H_2(Y)$ which together with f forms a $\mathbb{Z}$-basis of $H_2(Y)$. Let $H\cdot r = m$, $E\cdot r = n$, then n is an integer and m is an *odd* integer, the latter by Poincaré duality.

The reflexive sheaf $\mathcal{O}_X(D)$ is invertible away from C, so there is a Cartier divisor A on Y with a surjective map of sheaves

$$f^*(\mathcal{O}_X(D)) \to \mathcal{O}_Y(A)$$

the kernel of which is supported on the exceptional locus. In other words, as $\mathbb{Q}$-Cartier divisors,

$$A = f^*D + \beta E$$

with β rational. Then however $H + 2\beta E$ is 2-divisible in $H^2(Y)$, so

$$(H + 2\beta E)\cdot f = -2\beta \in 2\mathbb{Z}$$

and

$$(H + 2\beta E)\cdot r = m + 2\beta n \in 2\mathbb{Z}.$$

But this is a contradiction, for if the first integer is even, the second must be odd as m is odd. This proves the first statement.

Let now $r \in H_2(Y)$ denote a class which, together with l, gives a $\mathbb{Z}$-basis of $H_2(Y)$. The intersection numbers $H\cdot r$, $E\cdot r$ must be odd integers by Poincaré duality. But then the integers $(H+E)\cdot l$, $(H+E)\cdot r$ are both even and using Poincaré duality again, the second statement follows.

LEMMA 2·3. *$E\cdot H = kl \in H_2(Y,\mathbb{Z})$ for a positive integer k.*

Proof. By construction, E and H intersect in a number of fibres. This number is given by the intersection number $(2D)\cdot C$ on X, positive as D is ample.

Conclusion of the proof of Proposition 2·1. All that remains to do is to copy [**12**, 9·2]. By Kawamata–Viehweg, $h^0(X, nD) = \chi(X, nD)$ as D is ample. Next, if n is even then clearly $\chi(X, nD) = \chi(Y, \frac{1}{2}nH)$ and Riemann–Roch for the latter together with the obvious $D^3 = \frac{1}{8}H^3$, $c_2(X)\cdot D = \frac{1}{2}c_2(Y)\cdot H$ gives the formula. If n is odd, then by Lemma 2·2, $\frac{1}{2}(E+nH)$ is Cartier on Y. Moreover, $f_*\mathscr{O}_Y(\frac{1}{2}(E+nH)) = \mathscr{O}_X(nD)$ and $R^if_*\mathscr{O}_Y(\frac{1}{2}(E+nH)) = 0$ for $i>0$. So by Leray and Riemann–Roch on Y,

$$\begin{aligned}\chi(X,nD) &= \chi\left(Y,\tfrac{1}{2}(E+nH)\right)\\ &= \tfrac{1}{48}(E+nH)^3 + \tfrac{1}{24}c_2(Y)\cdot(E+nH)\\ &= \tfrac{1}{48}n^3H^3 + \tfrac{1}{16}nE^2\cdot H + \tfrac{1}{48}\left(E^3+2c_2(Y)\cdot E\right) + \tfrac{1}{24}nc_2(Y)\cdot H.\end{aligned}$$

This simplifies to the formula in the statement, as

$$E^3 = K_E^2 = 8(1-g)$$

and using [**4**, 2·3],

$$c_2(Y)\cdot E = c_2(E) - E^3 = 4(1-g) - 8(1-g) = 4(g-1).$$

The formula can be rewritten in the slightly more attractive form

COROLLARY 2·4.

$$h^0(X,nD) = \tfrac{1}{6}(n^3-n)D^3 + nh^0(D) + \tfrac{1}{8}nk\,I\;(n\equiv 0 \bmod 2).$$

3. *Classification*

The aim of this section is to classify varieties X with the properties stated at the beginning of Section 2, following Fletcher [**6**, III·9].

As explained in detail by Fletcher in [**6**, III·9], Riemann–Roch formulae can be used to generate examples of varieties in weighted projective spaces as follows: fix a set of constants $\{D^3, h^0(D), k\}$ and form the power series

$$P(t) = \sum_{n=0}^{\infty} h^0(X,nD)\,t^n$$

using the formula. If the triple $\{D^3, h^0(D), k\}$ comes from a variety

$$X = X_{d_1,\dots,d_c} \subset \mathbb{P}^m[w_0,\dots,w_m]$$

then this series equals the Hilbert series of the graded ring

$$R_D = \bigoplus_{n=0}^{\infty}\Gamma(X,\mathscr{O}_X(nD)).$$

It is well-known that this latter series has the closed form

$$P_{X,D}(t) = \frac{\prod_{i=1}^{c}(1-t^{d_i})}{\prod_{i=0}^{m}(1-t^{w_i})}.$$

Conversely, if $P(t)$ can be written in the closed form $P_{X,D}(t)$, then the constants $\{d_i, w_j\}$ can be read off and the resulting family of complete intersections *may* possess all the required properties.

In the first Appendix, I show that there is a finite set of triples $\{D^3, h^0(D), k\}$ satisfying the conditions. Details of the finite search are also in the Appendix; one obtains a list of possible configurations. An interesting feature of the list is that there are no codimension four complete intersections at all. Some of these configurations need to be discarded: the numerical conditions are not sufficient for the existence of quasi-smooth complete intersections.

Table 1.

The variety X	D^3	k	g	$\chi(Y)$	The other contraction on Y	Condition (3)
$X_{6,10} \subset \mathbb{P}^5[2^4, 3, 5]$	$\frac{1}{4}$	15	31	-176	Type II to sing. $Z_{3,5} \subset \mathbb{P}^5[1^5, 3]$	
$X_{4,6,6} \subset \mathbb{P}^6[2^5, 3^2]$	$\frac{1}{2}$	18	28	-148	K3 fibration over $\mathbb{P}^1$	$\surd$
$X_{12} \subset \mathbb{P}^4[1, 2^2, 3, 4]$	$\frac{1}{4}$	3	4	-144	Type II to sing. $Z_6 \subset \mathbb{P}^4[1^4, 2]$	$\surd$
$X_{14} \subset \mathbb{P}^4[1, 2^3, 7]$	$\frac{1}{4}$	7	15	-240	Type II to $Z_7 \subset \mathbb{P}^4[1^4, 3]$	
$X_{6,8} \subset \mathbb{P}^5[1, 2^3, 3, 4]$	$\frac{1}{2}$	6	7	-120	Type II to sing. $Z_{3,4} \subset \mathbb{P}^5[1^5, 2]$	$\surd$
$X_{4,10} \subset \mathbb{P}^5[1, 2^4, 5]$	$\frac{1}{2}$	10	16	-196	Type II to sing. $Z_{2,5} \subset \mathbb{P}^4[1^5, 2]$	
$X_{6,6} \subset \mathbb{P}^5[1, 2^4, 3]$	$\frac{3}{4}$	9	10	-120	Type II to sing. $Z_{3,3} \subset \mathbb{P}^5$	$\surd$
$X_{4,4,6} \subset \mathbb{P}^6[1, 2^5, 3]$	1	12	13	-128	Type II to sing. $Z_{2,2,3} \subset \mathbb{P}^6$	$\surd$
$X_{12} \subset \mathbb{P}^4[1^2, 2^2, 6]$	$\frac{1}{2}$	2	2	-252	K3 fibration over $\mathbb{P}^1$	$\surd$
$X_8 \subset \mathbb{P}^4[1^2, 2^3]$	1	4	3	-168	K3 fibration over $\mathbb{P}^1$	$\surd$
$X_{4,6} \subset \mathbb{P}^5[1^2, 2^4]$	$\frac{3}{2}$	6	4	-132	K3 fibration over $\mathbb{P}^1$	$\surd$
$X_{4,4,4} \subset \mathbb{P}^6[1^2, 2^5]$	2	8	5	-112	K3 fibration over $\mathbb{P}^1$	$\surd$

Theorem 3·1. *The complete intersection Calabi–Yau threefolds in codimension at most four, with a smooth curve of cA_1-singularities of genus at least two and $\mathcal{O}_X(1)$ a primitive $\mathbb{Q}$-Cartier divisor, general in moduli with these properties, are listed in Table 1*

Proof. The search detailed in the Appendix gives a 'raw list' of possible configurations, some of which need to be discarded as they do not give quasi-smooth varieties. [**6**, III·3·6–7], or analogous methods in the case of codimension three, prove that the general complete intersection is quasi-smooth for all families listed above.

As the varieties are complete intersections, $h^1(\mathcal{O}_X) = 0$ and rk Pic $(X) = 1$. Well-formedness is easily checked, so adjunction holds and gives $K_X \sim 0$. All singularities are quotient singularities; using the techniques discussed in [**6**, III·3·19], it is easily seen that in all cases and on all relevant singular strata they are $\frac{1}{2}(0, 1, 1)$ or cA_1 along a smooth curve, the genus g of which is easily computed. The Euler characteristic of Y is given by $\chi(Y) = \chi(X) - 3\,(g-1)$ using additivity of χ and $\chi(E) = 4\,(1-g)$. $\chi(X)$ can be computed from the formula in [**8**, section 2].

The entries in the last two columns will be checked in the final section.

Remark 3·2. Most of these varieties have of course occured in the literature before, although in a non-systematic way (see [**4**, **8**, **17**]); and in particular computer-generated lists such as Klemm's [**9**]. The last author ignores the possibility that one weight w_i can actually be *larger* than a degree d_i for codimension two complete intersections, cf. [**6**, III·9·14], so his list is not complete.

Remark 3·3. In the second Appendix, I make some remarks about non-complete intersection varieties whose existence is predicted by this method.

Remark 3·4. The clause about the varieties being general in moduli refers to the following: the method used above, based on the Hilbert series, cannot distinguish between the varieties $X_8 \subset \mathbb{P}^4[1^2, 2^3]$ and $X_{4,8} \subset \mathbb{P}^5[1^2, 2^3, 4]$ where the quartic polynomial does not involve the degree four variable. However, a general deformation of the latter is clearly an octic in $\mathbb{P}^4[1^2, 2^3]$. This phenomenon occurs with many of the varieties above.

Remark 3·5. The method of course supplies an explicit list of complete intersection Calabi–Yau threefolds in codimension at most four, with a smooth curve of cA_1-singularities of genus zero or one. As these are excluded by Condition (1) of Theorem 1·1, I omit this list; the interested reader will find it in [**16**].

4. *Contractions and automorphisms*

To complete the analysis of the examples, one needs to check conditions (2) and (3) of Theorem 1·1.

The penultimate column of Table 1 describes geometrically the nef cone of the smooth model Y which has Picard number 2, so its nef cone has precisely two extremal faces. One face is of course generated by H. The other face is expected to correspond to a different contraction: if the face is rational and generated by the nef (effective) Cartier divisor L, then for large m the morphism $g = \varphi_{|mL|} : Y \to Z$ maps Y to a Calabi–Yau Z with canonical singularities or a lower dimensional variety.

The singular X is a subvariety of a weighted projective space $\mathbb{P}$, which is a toric variety (I use the language of toric varieties [**7**]). This ambient variety can be partially resolved by a suitable blowup $\tilde{f} : \tilde{\mathbb{P}} \to \mathbb{P}$ which restricts to the resolution $f : Y \to X$ (cf. [**2**, **3**]). $\tilde{\mathbb{P}}$ is also a toric variety and it has a two-dimensional space of curves. So in the closed cone of curves $\overline{NE}(\tilde{\mathbb{P}})$, there are two extremal rays. The corresponding curves are represented by toric strata. Denote by $\tilde{g} : \tilde{P} \to \bar{P}$ the other contraction. If $\tilde{g}$ contracts a numerical class of curves that is represented in the subvariety Y, then $g = \tilde{g}\,|_Y$ gives the new contraction on Y.

The structure of the toric contraction is described by

THEOREM 4·1 (Reid [**11**]). *Let $\tilde{P}$ be an n-dimensional projective $\mathbb{Q}$-factorial toric variety. Let w be an $(n-1)$-dimensional cone in its fan such that the corresponding closed toric stratum l_w gives an extremal ray. Let w be the intersection $w = \sigma_n \bigcap \sigma_{n+1}$ of two different n-cones generated by the primitive vectors $\{f_1, \ldots, f_n\}$, $\{f_1, \ldots, f_{n-1}, f_{n+1}\}$. Then there is a relation*

$$\sum_{i=1}^{n+1} a_i f_i = 0$$

with $a_{n+1} = 1$, $a_n > 0$. Assume that a_i is negative for $1 \leqslant i \leqslant \delta$, zero for $\delta+1 \leqslant i \leqslant \beta$ and positive for $\beta+1 \leqslant i \leqslant n+1$, where of course $0 \leqslant \delta \leqslant \beta \leqslant n-1$. There is a contraction morphism $\tilde{g} : \tilde{P} \to \bar{P}$ contracting precisely those curves which are numerically equivalent to l_w. If $\tilde{g}\,|_A : A \to B$ denotes the locus where $\tilde{g}$ is not an isomorphism, then A, B are irreducible closed strata, A corresponds to the cone spanned by $\{f_1, \ldots, f_\delta\}$. One has $\dim A = n - \delta$ *and* $\dim B = \beta - \delta$.

This result can be used to check

PROPOSITION 4·2. *For every variety in Table 1, the contraction $\tilde{g}$ restricted to Y gives a nontrivial contraction $g\colon Y \to Z$, with the Type and image variety indicated. In particular, all the families satisfy condition* (2) *of Theorem* 1·1.

Proof. The proof is a not very difficult if somewhat tedious case-by-case check. I give the details of the calculation for two cases.

$X_{6,10} \subset \mathbb{P}^5[2^4, 3, 5]$. The ambient toric variety P is given in the lattice $N = \mathbb{Z}^5$ by the fan spanned by proper subsets of the vectors

$$e_0 = (-1,-1,-1,-4,1), \quad e_1 = (1,0,0,0,0), \quad e_2 = (0,1,0,0,0), \quad e_3 = (0,0,1,0,0),$$
$$e_4 = (0,0,0,1,1), \quad e_5 = (0,0,0,1,-1).$$

The partial resolution is obtained by barycentric subdivision using the extra vector

$$e_6 = \tfrac{1}{2}(e_4 + e_5) = (0,0,0,1,0).$$

The cone $w = (e_0 e_1 e_2 e_4)$ represents an extremal ray. The relation between the rays in the star of this cone is $-e_4 + e_0 + e_1 + e_2 + e_3 + 5e_6 = 0$. The contraction $\tilde{g}$ contracts the divisor given by the ray e_4 to the point $(0{:}0{:}0{:}0{:}0{:}1) \in \mathbb{P}^5[1^5, 3]$, where the latter is the toric variety spanned by the remaining edges.

The intersection of Y with the exceptional locus of $\tilde{g}$ is a divisor, contracted to a point, so $\tilde{g}$ restricts as a contraction of Type II to the Calabi–Yau hypersurface Y. The image Z of Y under this contraction is $Z_{3,5} \subset \mathbb{P}^5[1^5, 3]$, where the degree three equation does not involve the degree three variable. So Z is singular at the point $(0{:}0{:}0{:}0{:}0{:}1)$ of the weighted projective space.

$X_{4,6} \subset \mathbb{P}^5[1^2, 2^4]$. In this case the fan is given by

$$e_0 = (-1,-2,-2,-2,-2), \quad e_1 = (1,0,0,0,0), \ldots, \quad e_5 = (0,0,0,0,1)$$

in the lattice $N = \mathbb{Z}^5$, together with the vertex $e_6 = \frac{1}{2}(e_0 + e_1) = (0,-1,-1,-1,-1)$. The new extremal ray corresponds to the cone $w = \langle e_1, e_2, e_3, e_4 \rangle$ with relation $e_2 + e_3 + e_4 + e_5 + e_6 = 0$. Now $\alpha = 0$ and $\beta = 1$, so the exceptional locus is the whole $\tilde{P}$ and the image of the contraction is $\mathbb{P}^1$. This contraction restricts to the threefold Y as a K3 fibration over $\mathbb{P}^1$.

Remark 4·3. Notice that in two cases, Z is an interesting degeneration of the quintic threefold: the general deformation of the singular varieties $Z_{3,5} \subset \mathbb{P}^5[1^5, 3]$ or $Z_{2,5} \subset \mathbb{P}^4[1^5, 2]$ is a smooth quintic.

In the final Proposition, I check the last condition of Theorem 1·1.

PROPOSITION 4·4. *Let X be a general variety of any of the families in Table 1, marked with a tick in the final column. Then condition* (3) *of Theorem* 1·1 *is indeed satisfied for X.*

Proof. For the families $X_8 \subset \mathbb{P}^4[1^2, 2^3]$ and $X_{12} \subset \mathbb{P}^4[1^2, 2^2, 6]$, this was already proved in [**15**]. In all the remaining cases, the claim is that for general X, the automorphism group is trivial; this can be checked case-by-case, by explicit calculations. I do the computation in two of the cases again; the rest is done following a similar pattern.

$X_{12} \subset \mathbb{P}^4[1, 2^2, 3, 4]$. After a change of variables, the general equation is

$$F_8 = t^3 + f_8(x, y_i, z)\, t + (z^4 + f_6(x, y_i)z^2 + f_9(x, y_i)z + f_{12}(x, y_i)) = 0.$$

Assume that the automorphism σ given by

$$\begin{aligned} \sigma(x) &= \beta x, \\ \sigma(\vec{y}) &= A\vec{y} + Bx^2, \\ \sigma(z) &= \gamma z + F\vec{y}x + \lambda x^3, \\ \sigma(t) &= \delta\, t + \epsilon xz + DS^2\vec{y} + E\vec{y}x^2 + \eta x^4. \end{aligned}$$

fixes the hypersurface X_{12} in question. Then it must fix the singular locus, the general curve $C \cong C_6 \subset \mathbb{P}^2[1, 1, 2]$. Such a curve has no projective automorphisms, as can be checked directly, so necessarily $A = I, \delta = 1, D = 0$. As there is no t^2 term in F_8, $\epsilon = \eta = 0$, $E = 0$. The term having degree zero in t contains no z^3 term, so $F = 0$ and $\lambda = 0$. Thus, one is left with

$$\begin{aligned} \sigma(x) &= \beta x, \\ \sigma(\vec{y}) &= \vec{y} + Bx^2, \\ \sigma(z) &= \gamma z, \\ \sigma(t) &= t. \end{aligned}$$

On one hand, f_{12} is really a general sextic polynomial in x^2 and the y_is, so it has no automorphisms and hence $B = 0$, $\beta^2 = 1$. On the other hand, the z^2 term must be fixed so $\gamma^2 = 1$. Finally the general polynomial will contain an x^9z-term so $\beta\gamma = 1$. So $(\beta, \gamma) = \pm(1, 1)$, this latter sign change being part of the $\mathbb{C}^*$-action defining the weighted projective space.

$X_{4,4,4} \subset \mathbb{P}^6[1^2, 2^5]$. After a change of variables, the equations are

$$\left\{ \begin{array}{ll} F_4 = \sum y_i^2 + f_4(x_i) & = 0 \\ G_4 = y_1 g_2(x_i, y_2, \ldots, y_5) + g_4(x_i, y_2, \ldots, y_5) & = 0 \\ H_4 = y_1 h_2(x_i, y_2, \ldots, y_5) + h_4(x_i, y_2, \ldots, y_5) & = 0 \end{array} \right\}.$$

The singular locus is the general genus five curve $C_{2,2,2} \subset \mathbb{P}^4$ with no projective automorphisms. So σ must act as

$$\begin{aligned} \sigma(\vec{x}) &= A\vec{x}, \\ \sigma(\vec{y}) &= \vec{y} + BS^2(\vec{x}). \end{aligned}$$

However, the fact that the quadratic y_i terms are invariant and that y_1^2 only appears in F_4 forces $\sigma(F_4) = F_4$ and this immediately implies $B = 0$. Then A must be an automorphism of the general quartic f_4 and thus $A = \pm I$.

Proof of Theorem 0·1. The period map is finite using the main result of [**14**] and a computation identical to that of [**15**, 2·7]. The degree is at least two by Theorem 1·1 and the results of this section.

Remark 4·5. In the remaining three cases

$$X_{14} \subset \mathbb{P}^4[1, 2^3, 7], \quad X_{4,10} \subset \mathbb{P}^5[1, 2^4, 5], \quad X_{6,10} \subset \mathbb{P}^5[2^4, 3, 5]$$

the generic automorphism group is (at least) $\mathbb{Z}/2\mathbb{Z}$ and the corresponding involution violates condition (3) of Theorem 1·1. This was checked in one case in [**15**, 2·4]; the other two cases behave in the same way. In particular, these families do not provide counterexamples to weak global Torelli.

Appendices

The search

The task is to find triples $\{D^3, h^0(D), k\}$ such that the power series

$$P(t) = \sum_{n=0}^{\infty} h^0(X, nD)t^n$$

formed using the Dimension Formula (2·4) gives the closed Hilbert function

$$P(t) = \frac{\prod_{i=1}^{c}(1 - t^{d_i})}{\prod_{i=0}^{m}(1 - t^{w_i})}$$

of the graded ring of a complete intersection variety $X = X_{d_1,\ldots,d_c} \subset \mathbb{P}^m[w_0, \ldots, w_m]$ of dimension three. Denote $d = D^3$, $h = h^0(D)$.

The basic recursive method is the following (cf. [**6**, III·9·3]):

(i) fix a triple $\{d, h, k\}$ and form the power series $P(t)$;

(ii) find the smallest nonzero non-constant term $n_r t^r$ of $P(t)$;

(iii) assume that n_r is positive; then n_r of the weights on the weighted projective space must be r. Add these to the list of w_i-s, multiply $P(t)$ by $(1-t^r)^{n_r}$ and go back to the second step, unless the number of weights exceeds eight, in which case stop;

(iv) suppose that n_r is negative. Then there must be $\mid n_r \mid$ relations of degree r among the relations defining X. Add these to the list of d_is, multiply the power series by $(1 - t^r)^{n_r}$ again and go back to the second step unless either the power series becomes 1, or the number of relations exceeds four, in which cases stop.

The procedure either stops after producing at most eight weights and four less relations, in which case one obtains a family of varieties that must be investigated further, or else there are too many weights or relations. In the latter case the triple is discarded.

LEMMA. *There are only finitely many triples $\{d, h, k\}$ that need to be considered.*

Proof. The codimension of X is assumed to be at most four, so there are no more than eight weights, and it is easily seen that at least two of them must be different from one. So $h \leqslant 6$.

The formula for $h^0(2D)$ shows that $l = 4d$ and $m = \frac{1}{4}k + d$ must be positive integers. The coefficient of t^2 in $(1-t)^h P(t)$ is $\frac{3}{2}h - h^2 + m$ and there are two cases: either this coefficient is negative, which gives an upper bound for m immediately, or it is non-negative in which case it gives the number of twos among the weights, so necessarily $h + (\frac{3}{2}h - h^2 + m) \leqslant 8$ and this is an upper bound for m. Finally k as positive, $l < 4m$. Consequently there are finitely many possible possible triples $\{l, h, m\}$ and so finitely many possible triples $\{d, h, k\}$.

Based on this result and the particular numerical values, a short computer program produces the results discussed in Section 3. The interested reader can find the code of the program in [**16**].

Non-complete intersection varieties

The Hilbert series $P(t)$ gives information about non-complete intersection varieties as well, this is discussed in detail in [**1**]. Choosing integers w_i in a suitable way, the polynomial $\prod_{i=0}^{m}(1-t^{w_i})P(t)$ gives the degrees of the defining equations and the higher syzygies of such $X \subset \mathbb{P}^m[w_0, \dots, w_m]$, necessarily of codimension at least three. (The varieties studied here are automatically projectively Gorenstein and projectively Cohen–Macaulay, so if they have codimension at most two then they are complete intersections.) The search gives a finite list of these varieties for fixed codimension; the list below gives all codimension three candidates, with the degrees of the defining equations indicated. (There is a large number of codimension four candidates.)

The variety X	d	k
$X_{\{5,5,6,6,6\}} \subset \mathbb{P}^6[1^2, 2^3, 3^2]$	$\frac{3}{2}$	2
$X_{\{4,5,5,6,6\}} \subset \mathbb{P}^6[1^2, 2^4, 3]$	$\frac{7}{4}$	5
$X_{\{4,4,4,5,5\}} \subset \mathbb{P}^6[1^3, 2^4]$	$\frac{13}{4}$	3

A simple condition for existence is to use the Cohen–Macaulay assumption and 'cut the varieties by four hyperplanes', i.e. to find a product $\prod_{i=0}^{3}(1-t^{w_{j_i}})P(t)$ which is a polynomial with positive coefficients, giving the Hilbert series of a finite-dimensional Artinian ring. This property can easily be checked in the above cases. It would be of some interest to see whether the methods developed by Reid–Altınok (cf. [**1**]), can be used to establish the existence of these varieties.

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