Lie Groups

Michaelmas 2009

Question Sheet 4

Ad and ad , classical groups and their Lie algebras

1. (This problem gives another approach to Example 6.2.) Consider the adjoint representation Ad of SU_2 on its Lie algebra $L(SU_2) \simeq \mathbb{R}^3$.

   i. Prove that $\exp\,(diag(\theta i, -\theta i))$ acts on $\mathbb{R}^3$ by rotation through an angle 2θ around some axis.

   ii. Prove that every $X \in L(SU_2)$ is of the form $c\ diag(\theta i, -\theta i)\, c^{-1}$ for some c in SU_2.

   iii. Deduce Ad ($\exp X$) acts on $\mathbb{R}^3$ by rotation through an angle of 2θ around the axis (Ad c)$(0, 0, 1)^t$.

   iv. Use this to prove that Ad $: SU_2 \to SO_3$ is surjective and has kernel $\{I, -I\}$.

2. Show that the center of $\mathbb{H}$ is $\mathbb{R} =< 1 >$ the real vector space spanned by 1. Show that $SU_2 = Sp_1$. Show that $L(Sp_1) =< i, j, k >$. Describe Ad for Sp_1.

3. *Polar decomposition.* Let P be the set of all positive definite symmetric (resp. Hermitian) matrices, i.e. all symmetric (resp. Hermitian) matrices $a \in GL_n$ such that $(x, ax) \geq 0$ for all $x \neq 0$. Prove that $P \times O_n \to GL_n\mathbb{R}$ (resp. $P \times U_n \to GL_n\mathbb{C}$) given by $(p, a) \mapsto pa$ is a bijection. [*Hint: if* $a \in GL_n\mathbb{R}$ *then* $aa^t \in P$*; so* $aa^t = b^2$ *for some* $b \in P$ *and* $b^{-1}a \in O_n$*.*]

4. Let $s, t \in \mathbb{R}$ and $X = X(t)$ be a matrix valued function. Consider
$$Y(s,t) = \exp\,(-sX)\frac{d}{dt}\exp sX.$$

   i. Show that $\frac{d}{ds}Y(s,t) = \exp\,(\mathrm{ad}\,(-sX))\,\frac{d}{dt}X$;

   ii. Hence show that
$$Y(1,t) = \int_0^1 \frac{d}{ds}Y(s,t)\,ds = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{ad}\ X)^k\,\frac{d}{dt}X;$$

   iii. Use this to compute $\frac{d}{dt}\exp X$.

   iv Find the derivative of $\exp\,(A + tB)$ at $\exp A$.