Notes of a Numerical Analyst

Is Everything a Rational Function?

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There's an old idea that I call the Kirchberger Principle, since it was expressed by Hilbert's student Paul Kirchberger in 1903. (It almost surely goes back further, and if you know a 19th-century source, I'd be glad to hear from you.)

> Since the only operations that can be carried out numerically are the four elementary operations of addition, subtraction, multiplication and division, it follows that we are only masters of more general functions insofar as we can replace them by rational functions, that is, represent them approximately.

This observation seems so basic that one could hardly doubt it. But it leads to a puzzle. If Kirchberger's Principle is valid, then numerical analysis should be more or less synonymous with rational approximation theory. This is patently not so. Why not? I've been thinking about this a good deal, and I'd like to offer three explanations.

The first is an observation about computers. The closer you look at actual machines, the harder it is to argue that +, −, *, / are the atoms from which other operations are composed. In practice, / is reduced to the first three by algorithms such as Newton's method. Moreover, very similar reductions are used for $\sqrt{\ }$, making it hard to justify any difference in status between / and $\sqrt{\ }$.

The second observation also has to do with how we, and our machines, actually compute. We *manipulate digits.* This goes well beyond +, −, *, /, for it requires comparisons and branches. For example, the first step in evaluating $\sin(x)$ may be to shift x to the interval $[0, 2\pi)$. So Kirchberger must be modified by the footnote that our approximations are not globally rational but *piecewise rational.*

The third observation is the most interesting mathematically. Suppose we do Newton's method, say, to evaluate $\sqrt{x}$:

$$t_{k+1} := \frac{1}{2}\left(t_k + \frac{x}{t_k}\right).$$

If $t_0 = 1$, the result after k steps is a *composite* rational function of x of the form $r_k(\cdots r_1(x)\cdots)$, and it has degree 2^{k-1}, not k. This is far from the usual setup in approximation theory, where we consider approximations in the class of all rational functions of a given degree n.

The more you dig about these questions, the further you find yourself from Kirchberger. In almost any calculation with real numbers, to get d digits of accuracy, we must do at least $O(d)$ work. For addition, $O(d)$ is enough. For multiplication, division or square root, it's $O(d \log d)$. A process involving Newton's method would seem to multiply the cost by another $\log d$, but that can be avoided if you adjust the precision as you go.

So in computing with real numbers, no operations are atomic: they all require more work as you demand more accuracy. And once you realize this, the traditional distinction between finite and infinite processes fades away. A linear system of equations $Ax = b$ is the archetypical *finite* problem of numerical linear algebra, solvable in a finite number of steps, whereas an eigenvalue problem $Ax = \lambda x$ is *infinite,* requiring an iteration. But in the end they both cost $O(d \log d)$.

FURTHER READING

[1] R. P. Brent and P. Zimmermann, *Modern Computer Arithmetic,* Cambridge, 2010.
[2] E. Gawlik and Y. Nakatsukasa, Approx- imating the pth root by composite rational functions, *J. Approx. Theory*, 2021.
[3] D. Harvey and J. van der Hoeven, Integer multiplication in time $O(n \log n)$, *Annals of Math.,* 2021.
[4] P. Kirchberger, Über Tchebychefsche Annäherungsmethoden, *Math. Ann.*, 1903.

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