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Surprises of the Faraday Cage

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Nearly everyone has heard of the Faraday cage effect. So when I needed to learn about it, I assumed it would be a matter of looking in some standard physics books, maybe the ones I'd studied as an undergraduate. This was the beginning of a journey of surprises.

The Faraday cage effect involves shielding of electrostatic and electromagnetic fields. A closed metal cavity makes a perfect shield, with zero fields inside, and that *is* in the textbooks. Faraday's discovery of 1836 was that fields are nearly zero inside a wire mesh, too. You see this principle applied in your microwave oven, whose front door contains a metal screen with small holes. The screen keeps the microwaves in, while allowing light, with its much smaller wavelength, to pass through.

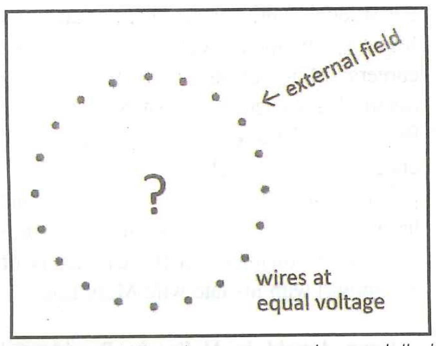
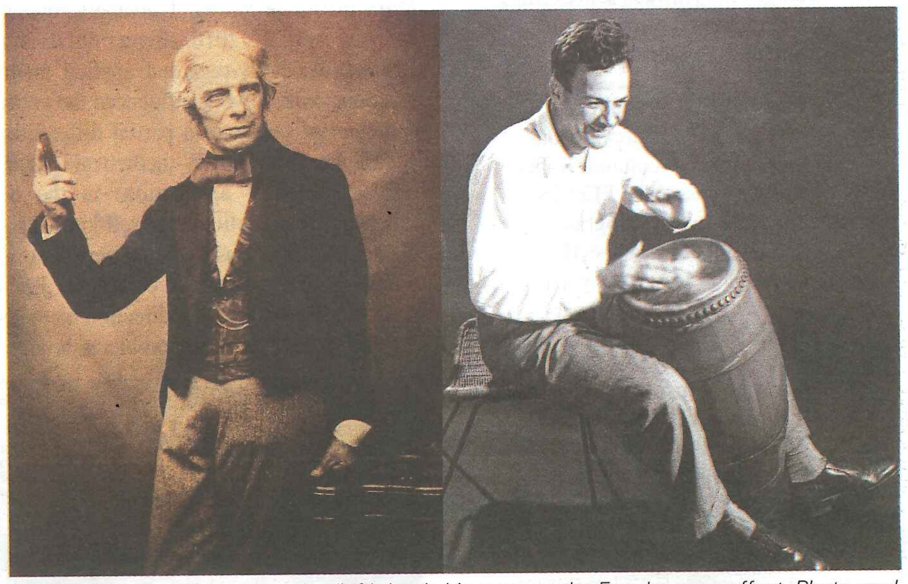


Figure 1. *A Faraday cage can be modelled by a set of dots (cross-sections of wires) spaced around a circle, with equal potential on each. If a potential is applied outside the cage, how close to zero is the field (potential gradient) inside? Figure adapted from [1].*

The essence of the matter can be captured by a two-dimensional model (see Figure 1), where the cage is approximated by a circle or a line of dots representing cross-sections of wires all at the same voltage (connected somewhere in the third dimension). To keep things simple, we focus on electrostatic fields – the Laplace equation.

Let me explain how I got interested in this problem. André Weideman and I were finishing a survey of the trapezoidal rule for periodic analytic functions, which we'd been working on for eight years [5]. We knew the mathematics of that problem: if f is analytic and periodic and you add up sample values at equispaced points, you get an exponentially accurate approximation to its integral. Intuitively, sinusoidal oscillation in one direction corresponds to exponential decay in the direction at right angles in the complex plane. A contour integral estimate of Fourier coefficients exploits this decay to prove exponential accuracy.

To enrich our survey, I thought we should comment on the analogy between this mathematics of the trapezoidal rule and that of the Faraday cage. It seemed obvious the two must be related – it would just be a matter of sorting out the details.



English scientist Michael Faraday (left), lends his name to the Faraday cage effect. Photo credit: Wikimedia Commons. Richard Feynman (right), we were surprised to learn, got it wrong. Photo courtesy of the Archives, California Institute of Technology.

So I started looking in books and talking to people and sending emails. In the books, nothing! Well, a few of them mention the Faraday cage, but rarely with equations. And from experts in mathematics, physics, and electrical engineering, I got oddly assorted explanations. They said the skin depth effect was crucial, or this was an application of the theory of waveguides, or the key point was Babinet's principle, or it was Floquet theory, or "the losses in the wires will drive everything…"

And then at lunch one day, colleague $n+1$ told me, it's in the *Feynman Lectures* [2]!

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Faraday Cage

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And sure enough, Feynman gives an argument that appears to confirm the exponential intuition exactly. He sets up a model of an array of charged wires (see Figure 2) and shows with simple formulas that electrostatic shielding is exponentially effective for just the reason I had imagined: because periodic in one direction means exponential decay at right angles. He writes:

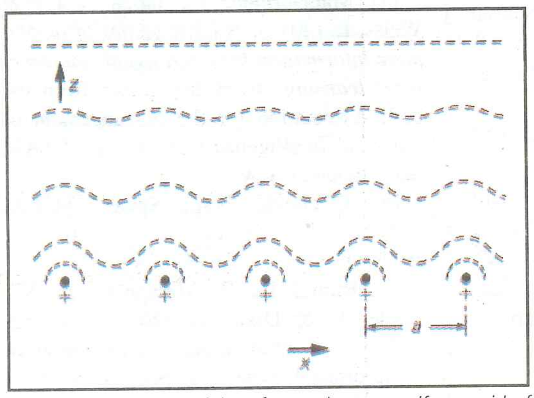

Figure 2. *Equipotential surfaces above a uniform grid of charged wires. Excerpted from The Feynman Lectures on Physics, Volume II by Richard P. Feynman. Copyright © 1964. Available from Basic Books, an imprint of Perseus Books, LLC, a subsidiary of Hachette Book Group, Inc.*

> The method we have just developed can be used to explain why electrostatic shielding by means of a screen is often just as good as with a solid metal sheet. Except within a distance from the screen a few times the spacing of the screen wires, the fields inside a closed screen are zero.

Now Feynman is a god, the ultimate cool genius. It took me months, a year really, to be confident that the great man's analysis of the Faraday cage, and his conclusion of exponential shielding, are completely wrong.

The error is that Feynman's wires have constant *charge*, not constant *voltage*. It's the wrong boundary condition! I think that Feynman, like me and most others beginning to think about this problem, must have assumed that the wires may be taken to have zero radius. The trouble is, a point charge makes sense, but a point voltage does not. (Dirichlet boundary conditions for the Laplace equation can only be applied on sets of positive capacity.) Since the correct boundary condition cannot be applied at points, I'm guessing Feynman reached for one that could, intuiting that it would still catch the essence of the matter. This is a plausible intuition, but it's wrong.

Feynman's calculation is arithmetically correct: an infinite array of equal point charges generates a far field that is exponentially close to uniform. However, this isn't the configuration of Faraday shielding. In fact, the point charge model doesn't include a source to be shielded, or a wavelength. As soon as you realize these things, if you are a numerical analyst like me, you want to compute some solutions of the true PDE problem, like those shown in Figure 3.

The computations reveal two big facts. First of all, *the radius of the wires matters*. As $r \to 0$, the shielding goes away. This, we now realize, must be why your microwave oven door has so much metal in it, and is not just a sheet of glass with a thin wire grid.

Secondly, *the shielding is linear in the gap size, not exponential*. If it were exponential, the field strength inside the cavity would square when you halve ε, the gap between the wires. In fact, it just cuts in half. This may be why your cell phone often works in surprising places, like inside an elevator. The analysis shows that in the limit $r \ll \varepsilon \ll 1$, the field scales as $\varepsilon \log r$.

I have started to speak of "we." As the study progressed, I knew I had to get more serious mathematically. This was the beginning of a happy collaboration with Jon Chapman and Dave Hewett, who share a hallway with me at Oxford. As a threesome, with varied backgrounds, we talked to more people and learned more. For example, we learned that Maxwell in his treatise from the 1870s considered an infinite array of wires and got the physics right, including the logarithmic dependence on radius [4]. Why has Maxwell's work been forgotten?

Most importantly, Chapman and Hewett developed an analysis in which a wire cage is modeled by a continuum boundary condition. Intuitively, it cannot be necessary to describe your microwave oven door hole-by-hole; there must be a homogenized boundary condition that has the same effect. Using multiple-scales analysis, Chapman and Hewett found this boundary condition, involving the jump in the normal derivative of the potential, which makes precise the idea that a metal screen behaves like a continuous substance that is not quite a metal. A physical interpretation involves energy minimization in surfaces of restricted electric capacity. The figures in [1] show strikingly that the homogenized model and an energy minimization calculation match the true behavior as found in the numerical simulations, and Hewett and Ian Hewitt have gone on to extend the analysis to electromagnetic waves [3].

In closing, I want to reflect on some of the curious twists of this story, first, by mentioning three lessons:

L1. There are gaps out there. If you find something fundamental that nobody seems to have figured out, there's a chance that, in fact, nobody has.

L2. Analogies are powerful. I would never have pursued this problem had I not been determined to understand the mathematical relationship between the Faraday cage and the trapezoidal rule.

L3. Referees can be useful. Thank you, anonymous man or woman who told us the Faraday cage section in our trapezoidal rule manuscript wasn't convincing! We removed those embarrassing pages, and proper understanding came months later.

And then three questions:

Q1. How can arguably the most famous effect in electrical engineering have remained unanalyzed for 180 years?

Q2. How can a big error in the most famous physics textbook ever published have gone unreported since 1964?

Q3. Somebody must design microwave oven doors based on laboratory measurements. Where are these people?

References

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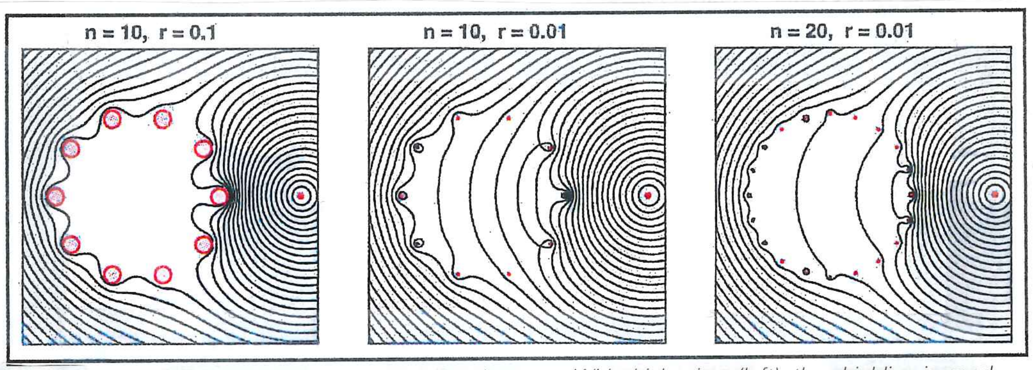

Figure 3. *Computed equipotential lines in a Faraday cage. With thick wires (left), the shielding is good. With thin wires (center), the shielding is weak. As the gap between the wires is reduced (right), the shielding improves only linearly, not exponentially. Figure adapted from [1].*