Laplace Equation Checklist

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u is a real function on a domain $\Omega \subseteq \mathbb{R}^n$ with boundary $\partial\Omega$,
u_n = outward normal derivative on $\partial\Omega$, $r = |x| = (x_1^2 + \cdots + x_n^2)^{1/2}$.

nD

Laplace equation	$\Delta u = 0$
Harmonic function	A solution of the Laplace equation
Potential theory	The study of the Laplace equation
Classical applications	Heat or diffusion equilibria, electrostatics, ideal fluid flow, membranes
Separation of variables	In a box, you get trigonometric in some directions, exponential in others
Fundamental solution	$u = r^{2-n} \cdot \Gamma(n/2)\pi^{-n/2}/(4-2n)$, except $u = \log(r)/2\pi$ for $n = 2$
Method of fundamental solns	Numerical method based on approx by linear combs. of fundamental solns
Green function	Like a fundamental solution, but for bounded Ω with $u = 0$ on $\partial\Omega$
Maximum principle	$\max u$ is always attained on $\partial\Omega$ ($\min u$ too)
Mean-value principle	u harmonic $\Leftrightarrow$ u(center of any sphere) = mean(u(values on the sphere))
Real-analyticity	A harmonic function is real-analytic (Taylor series in $x_1, \ldots, x_n$)
Liouville's theorem	A bounded harmonic function on $\mathbb{R}^n$ is constant
Schwarz reflection principle	If $u = 0$ on a portion of a (hyper)plane or sphere, it can be reflected across
Dirichlet problem	u specified on $\partial\Omega \Rightarrow \exists$ unique solution to $\Delta u = 0$
Dirichlet principle	This solution is the minimizer of $\int_\Omega \nabla u \cdot \nabla u = \int_\Omega u_{x_1}^2 + \ldots + u_{x_n}^2$
Neumann problem	u_n specified on $\partial\Omega$ with $\int_{\partial\Omega} u_n = 0$, Ω connected $\Rightarrow \exists\,!$ soln up to a const
Poisson equation	$\Delta u = f$

3D

Spherical coordinates	$\Delta u = u_{rr} + 2r^{-1}u_r + (r^2\sin\varphi)^{-1}(u_\varphi \sin\varphi)_\varphi + (r^2\sin^2\varphi)^{-1}u_{\theta\theta}$
Cylindrical coordinates	$\Delta u = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz}$
Solid harmonics	Harmonic functions on a ball
Spherical harmonics	Their restrictions to a sphere

2D

Polar coordinates	$\Delta u = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}$
Poisson kernel	$P(x,y) = (1-\lvert x\rvert^2)/\lvert y-x\rvert^2$
Poisson formula	Solution of Dirichlet problem in unit disk: $u(x) = \int_{\lvert y\rvert=1} P(x,y)u(y)d\theta$
Complex analytic functions	u harmonic in simply-connected Ω $\Leftrightarrow$ $u = \mathrm{Re}(f)$ for some analytic f
Conjugate harmonic function	$v = \mathrm{Im}(f)$, unique up to a constant
Domains with holes	Above becomes $u = \mathrm{Re}(f) + \sum_k c_k \log\lvert x - x_k\rvert$, one log term for each hole
Conformal mapping	Laplace probs are invariant, so simply-conn. ones can be reduced to a disk

1D

Not much to say, since $u_{xx} = 0$ is just an ODE, with general solution $ax + b$.