Favorite Eigenvalue Problems

I'd like to offer you some of my favorite eigenvalue problems.

There's the Bay of Fundy in Canada. Where does it get those 16-meter tides you can read about in the *Guinness Book of Records*? Well, the turning of the earth against the moon provides a gravitational tap every 12 hours and 25 minutes, like pushing a child on a swing, and a natural mode of oscillation of the bay has a period of about 13 hours. So the giant tides come from resonance fed by pumping at a frequency near an eigenvalue.

Musical instruments are built upon eigenvalues. The air in a flute vibrates according to the wave equation with atmospheric pressure at each end, so you get sine waves of pressure at approximately linearly spaced eigenfrequencies. The regular spacing explains why a note sounds musical, and the sine shapes explain why the thumb key is located halfway along: When you open the key, you attenuate modes 1,3,5, . . . more than 2,4,6, . . . , and the pitch goes up an octave. In a clarinet, the pitch goes up an octave and a fifth. That's because the chamber is closed at the reed end, making the boundary condition Neumann instead of Dirichlet, so the lowest eigenmode is half a sine wave, the next is 1½ sine waves, and so on. Opening the thumb key attenuates ½ in favor of 1½ and triples the frequency.

FROM THE SIAM PRESIDENT

By Nick Trefethen

Any regular pattern you see in nature probably has something to do with an eigenvalue problem. Cloud stripes in the sky, ripples on the sand, droplets in a splash—the mathematics in common is that some eigenmodes grow faster than others, and the wavelength you see is one of the fastest.

Eigenvalues have interested pure mathematicians too, as the limerick reminds us:

A professor said, maybe I'm dumb,
But I can't hear the shape of a drum!
Then along came a theorem:
Nobody can hear 'em!
Now Professor Pi isn't so glum.

Eigenvalues of random matrices make a beautiful demonstration on the computer. Take an $n \times n$ matrix with independent entries of standard deviation $n^{-\frac{1}{2}}$, and you'll find that its eigenvalues fill the complex unit disk uniformly—exactly so in the limit $n \rightarrow \infty$. This is Girko's circular law, recently generalized by Tao and Vu. Symmetric matrices follow analogous laws for distributions of real eigenvalues, and these have been connected with nuclear physics.

Eigenvalues played a role in a great coincidence of scientific history. Physicists saw spectral lines in the light from stars; indeed, an unfamiliar line led to the discovery of helium. Later, Hilbert defined spectra of operators. Not until Schrödinger, decades on, was it understood that physicists' spectra were exactly a case of mathematicians' spectra, with each line corresponding to the difference in energy of two eigenstates. And it was these same spectral lines that led to the discovery of the red shift and the expanding universe.

Lasers couple this eigenvalue problem of atomic physics with another eigenvalue problem, involving light waves bouncing back and forth in a cavity. It's because an eigenvalue gap in the atom is tuned to an eigenvalue of the cavity that the device emits a coherent beam. AM radios are like lasers in reverse. Instead of a resonant cavity building up a big signal, a bit of which is allowed to escape to the outside, you have a weak signal from outside that is brought into a resonant circuit and greatly amplified. To change stations, you adjust the eigenvalue.

Most eigenvalue problems in physics are self-adjoint, or nearly so (I think at least twenty physics Nobel Prizes have come from phenomena related to eigenvalues), but high-Reynolds-number fluid flows are an exception because of the interplay of convection and diffusion. Non-self-adjointness makes eigenvalues physically problematic, and indeed the eigenvalues of these flows, which have been studied by mathematicians for a century, are misleading: When they lie in the left half of the complex plane, suggesting stability, the flow is often unstable in practice anyway.

Magnetohydrodynamics features even knottier non-self-adjoint eigenvalue problems, which may be one of the reasons fusion power has been elusive, and may also help explain why the magnetic field generated by the dynamo in the core of the earth switches direction seemingly at random. It is the trace of these random reversals in core samples from ocean bottoms that led to acceptance of the theory of continental drift, a process driven by yet another eigenvalue problem: the slow radioactive decay of nuclei from their metastable near-eigenstates, which pumps heat into the earth's core that must escape at the surface.

One day somebody may win a million dollars from an eigenvalue problem. If the zeros of the Riemann zeta function lie on the critical line, might the reason be that they are connected with eigenvalues of a hermitian operator? Many people think so, and the Riemann hypothesis is one of the Clay Institute's Millennium Prize Problems.

Let's finish with the story about the couple in the double bed with an electric blanket with independent his-and-hers controls. It's supposed to be a stable system governed by a nice 2×2 matrix, the negative of the identity, with eigenvalues −1 and −1, each partner adjusting a separate dial to warm up if too cold and cool down if too hot. But what if the wires accidently get crossed? The columns of the matrix get interchanged, and now the eigenvalues are −1 and +1. Another limerick summarizes the dire outcome, as predicted by linear eigentheory:

For their wedding, a couple acquired
A new blanket they sadly miswired.
The new husband fried;
Bitter cold killed the bride.
And so, side by side, they expired.