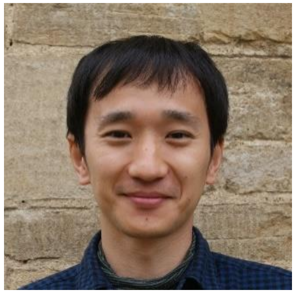
Vandermonde with Arnoldi

Nick Trefethen, University of Oxford

1. VANDERMONDE
2. MONOMIALS
3. ARNOLDI
4. EIGHT EXAMPLES

Paper submitted to *SIAM Review* with
Pablo Brubeck and Yuji Nakatsukasa



1. VANDERMONDE

$$p(x) = \sum_{k=0}^{n} c_k x^k$$

Interpolation or least-squares: $Ac \approx f$

$$\begin{pmatrix} 1 & x_1 & \cdots & x_1^n \\ 1 & x_2 & \cdots & x_2^n \\ \vdots & \vdots & & \vdots \\ 1 & x_m & \cdots & x_m^n \end{pmatrix} \begin{pmatrix} c_0 \\ \vdots \\ c_n \end{pmatrix} \approx \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix}$$

```
c = polyfit(x,f,n)
```

```
function c = polyfit(x,f,n)
A = x.^(0:n);
c = A\f;
```

Evaluation: $y = Bc$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} 1 & s_1 & \cdots & s_1^n \\ 1 & s_2 & \cdots & s_2^n \\ \vdots & \vdots & & \vdots \\ 1 & s_M & \cdots & s_M^n \end{pmatrix} \begin{pmatrix} c_0 \\ \vdots \\ c_n \end{pmatrix}$$

```
y = polyval(c,s)
```

```
function y = polyval(c,s)
n = length(c)-1;
B = s.^(0:n);
y = B*c;
```

These days the rectangular case is particularly interesting. Redundant bases, frames,…

2. MONOMIALS

$1, x, \ldots, x^n$ is exponentially ill-conditioned on $[-1,1]$ (on any domain except a disk)

```
x = chebfun('x');
cond(x.^(0:10))          ... 20,40
plot(x.^(0:40))
xx = chebpts(1000);
cond(xx.^(0:10))         ... 20,40
```

$\kappa \approx \left(1+\sqrt{2}\right)^n$ (Gautschi 1975)

Computational consequence: $n \gg 30$ never works.

x^n has numerical degree $O(\sqrt{n})$ on $[-1,1]$

Newman & Rivlin 1976

```
length(x^10)                 ... 20, 40, 80, 1000, 4000, 16000
length(chebpoly(16000))
```

Müntz-Szász theorem

3. ARNOLDI

Problem: $\{1, x, x^2, \ldots\}$ is ill-conditioned, so computations fail.

Solution: $\{1, x, x^2, \ldots\} = \{q_0, Aq_0, A^2 q_0, \ldots\}$ where $A = \mathrm{diag}(x)$. So do Arnoldi!

Idea of Arnoldi: instead of forming A^n then orthogonalizing, orthogonalize at each step.

Applied to $\{1, x, x^2, \ldots\}$, this is Stieltjes orthogonalization. A very old idea.

Austin et al., Betcke, Björck & Pereyra, Forsythe, Gautschi, Gragg, Hochman, Reichel, Saad, Stylianopoulos, undoubtedly many others.

This is a technique we should use routinely.
Not just “when we want to construct orthogonal polynomials.”

3. ARNOLDI, cont.

Arnoldi/Stieltjes applied to $\{1, x, x^2, \ldots\}$ constructs discrete orthogonal polynomials related to the monomials by a Hessenberg matrix H.

We now pass around H as well as a coefficient vector.

```
[d,H] = polyfitA(x,f,n)
```

```
function [d,H] = polyfitA(x,f,n)
m = length(x);
Q = ones(m,1);
H = zeros(n+1,n);
for k = 1:n
    q = x.*Q(:,k);
    for j = 1:k
        H(j,k) = Q(:,j)'*q/m;
        q = q - H(j,k)*Q(:,j);
    end
    H(k+1,k) = norm(q)/sqrt(m);
    Q = [Q q/H(k+1,k)];
end
d = Q\f;
```

$O(mn^2)$ flops, same as `polyfit`.

```
y = polyvalA(d,H,s)
```

```
function y = polyvalA(d,H,s)
M = length(s);
W = ones(M,1);
n = size(H,2);
for k = 1:n
    w = s.*W(:,k);
    for j = 1:k
        w = w - H(j,k)*W(:,j);
    end
    W = [W w/H(k+1,k)];
end
y = W*d;
```

$O(Mn^2)$ flops; `polyval` is $O(Mn)$.

$n =$ degree, $m =$ no. of sample pts, $M =$ no. of evaluation points

($O(mn)$ and $O(Mn)$ possible when x is real via Arnoldi → Lanczos, though we don't do this.)

4. EIGHT EXAMPLES

1. Degree n interpolation of $1/(1+25x^2)$ in Chebyshev pts
2. Degree n least-squares fit to sign(x) on $\left[-1,-\frac{1}{3}\right] \cup \left[\frac{1}{3},1\right]$
3. Degree 30 Chebyshev polynomial on a triangle in $\mathbb{C}$
4. Degree n Fourier extension fit of $1/(10-9x)$ on $[-1,1]$
5. Bivariate polynomial fit on a starfish domain
6. Conformal mapping via polynomial approximation of Green's function
7. Lightning Laplace solver
8. Stokes flow

Example 1: Degree n interpolation of $1/(1+25x^2)$ in Chebyshev points

interpolant →

error →

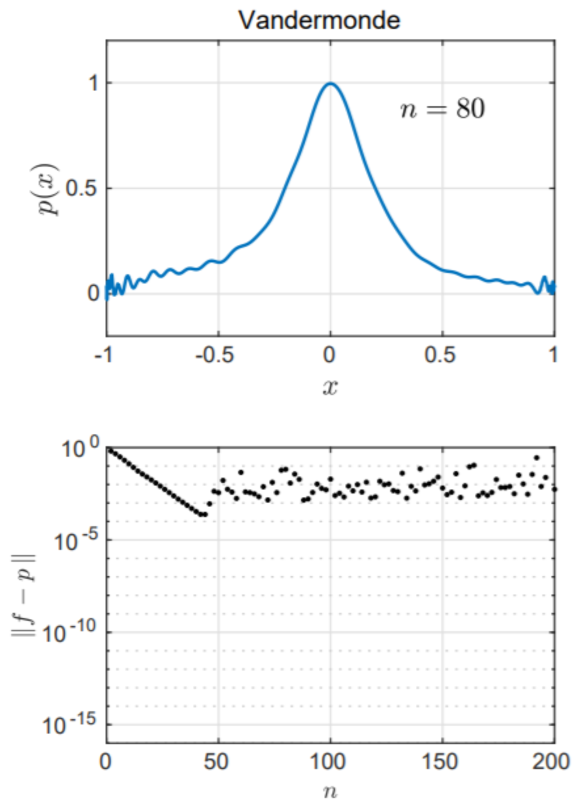


FIG. 2.1. *On the left, the degree n Chebyshev interpolant to $f(x) = 1/(1+25x^2)$ computed unstably by direct application of (1.2) and (1.3) via the codes* **polyfit** *and* **polyval** *for $n = 80$ (above) and its error for even values of n from 2 to 200 (below). (The results computed by the MATLAB versions of* **polyval** *and* **polyfit** *would be worse.) On the right, the same computations with the Arnoldi-based codes* **polyfitA** *and* **polyvalA**.

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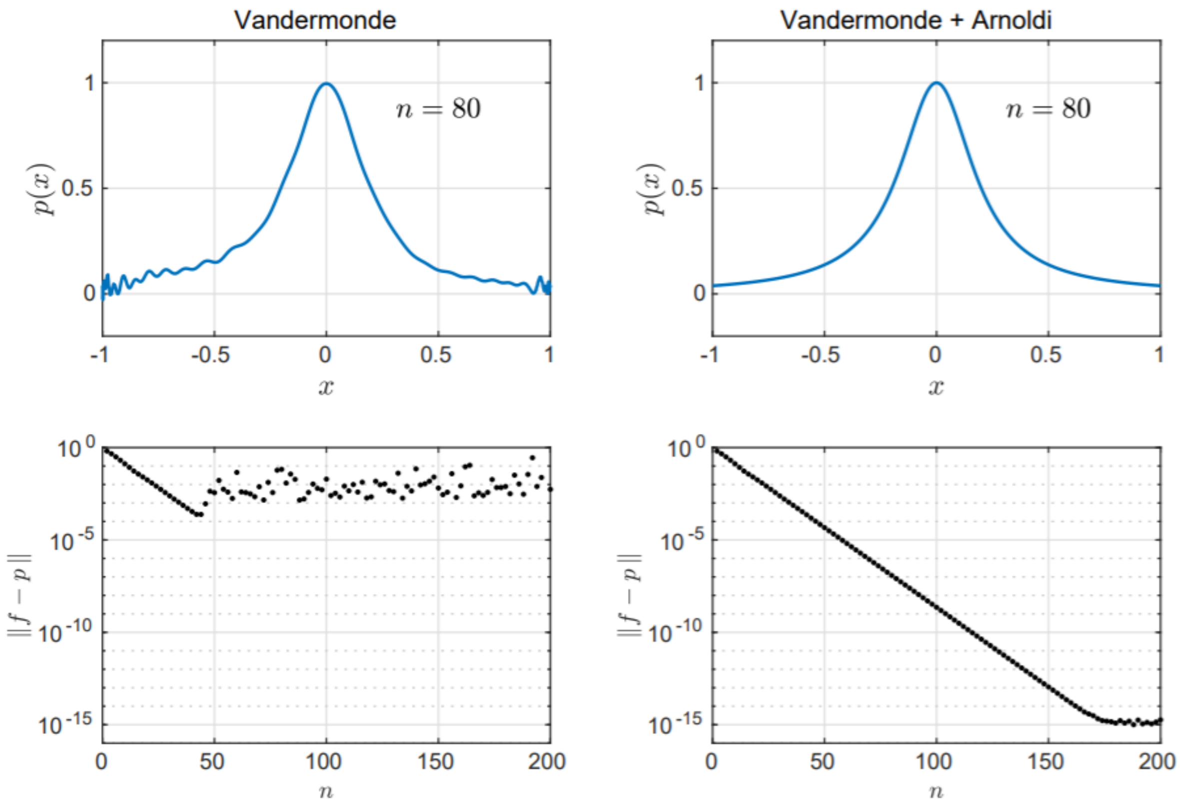

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Example 2: Degree n least-squares fit to sign(x) on $\left[-1,-\frac{1}{3}\right] \cup \left[\frac{1}{3},1\right]$

approximant →

error →

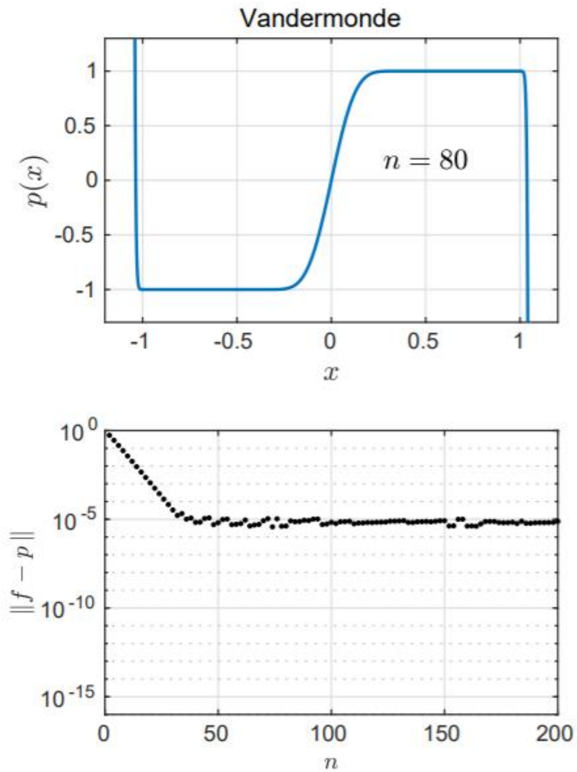

FIG. 3.1. *Images as in Fig. 2.1 but now for a least-squares problem: polynomial fitting to sign*(x) *on* 500 *equispaced points each in the two intervals* $[-1,-1/3]$ *and* $[1/3,1]$. *The unstable algorithm stagnates at* 5 *digits of accuracy, which is enough that to the eye, the computation appears successful.*

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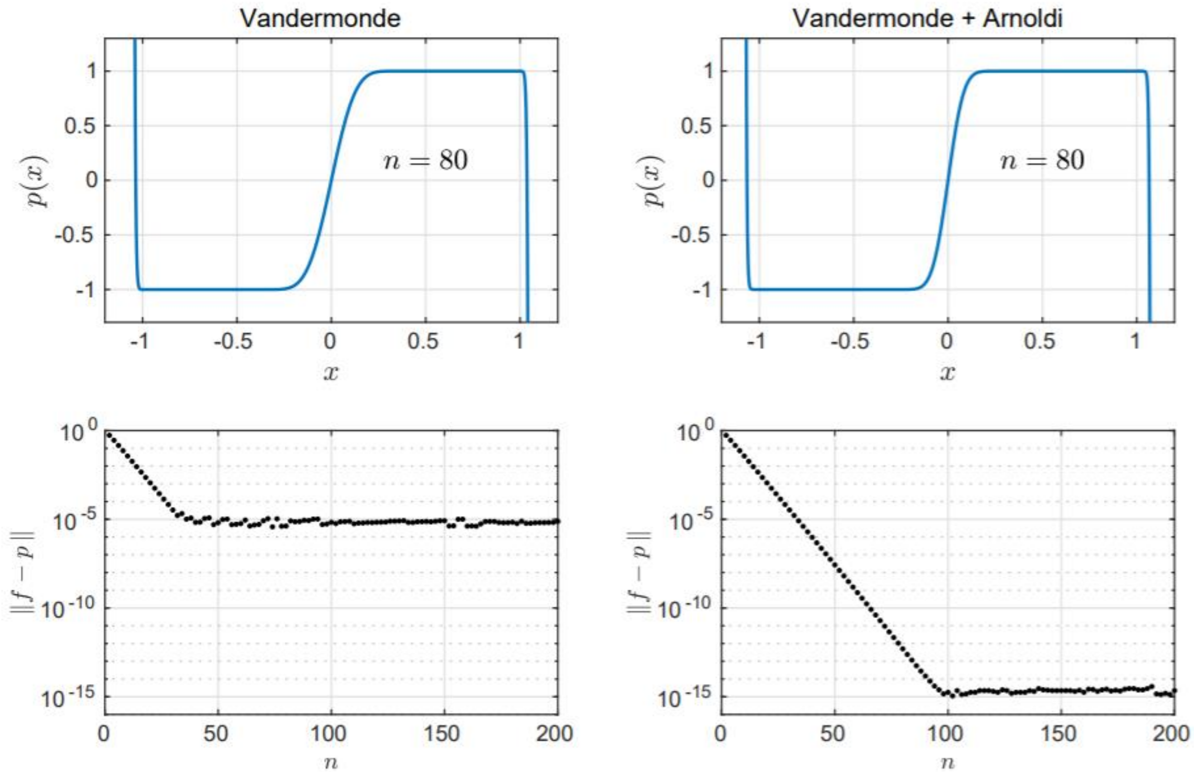

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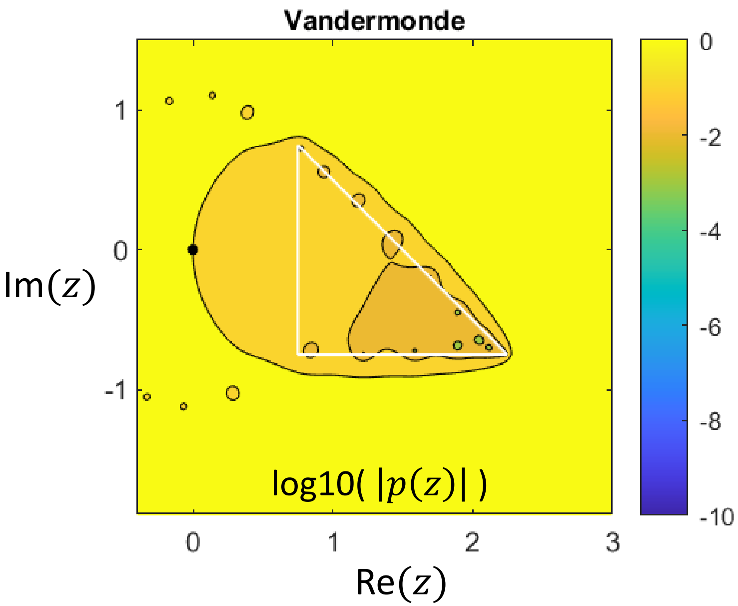

Minimal monic polynomial with $p(0) = 1$.

We use the Lawson algorithm
(iteratively reweighted least-squares).

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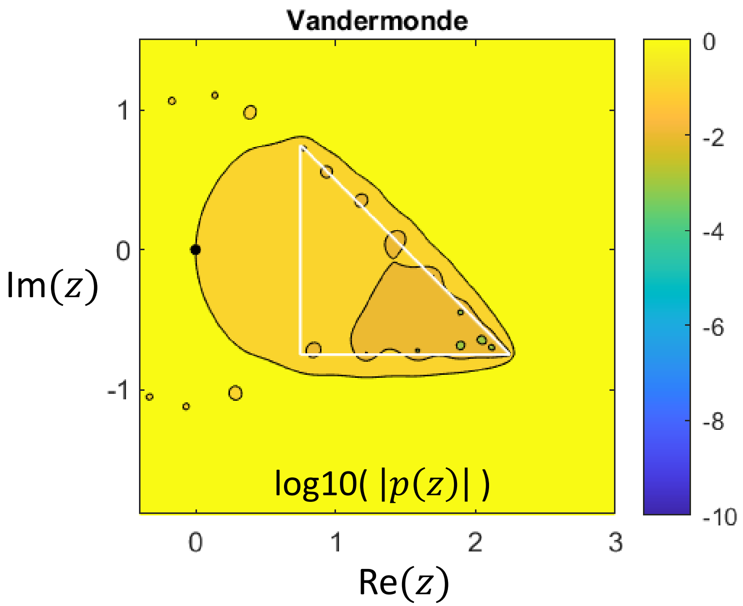

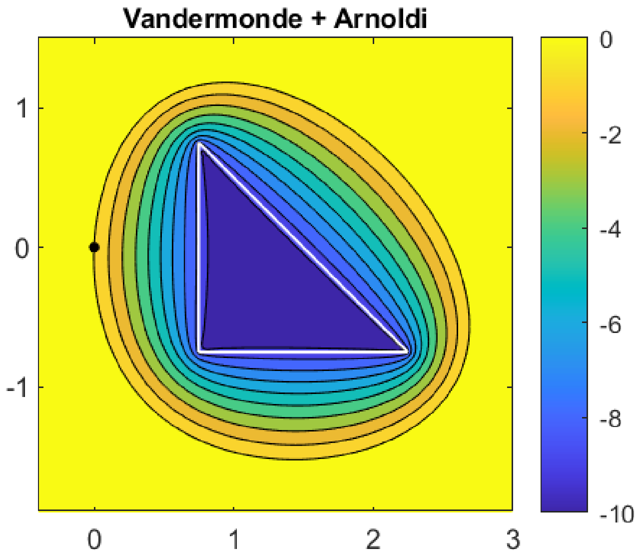

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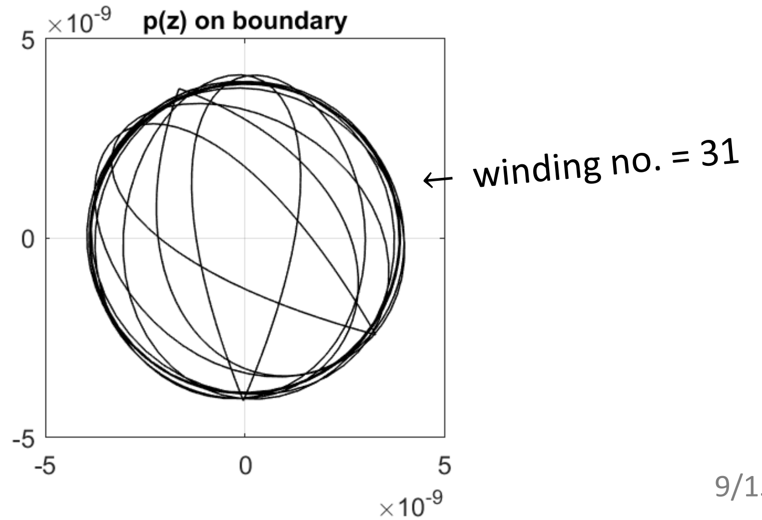

Example 4: Degree n Fourier extension fit of $1/(10-9x)$ on $[-1,1]$

Example from
Adcock + Huybrechs,
SIAM Review 2019

Key observation:

Fourier series on subinterval of $[-2,2]$

$\Updownarrow$

Laurent polynomial on subarc of unit circle

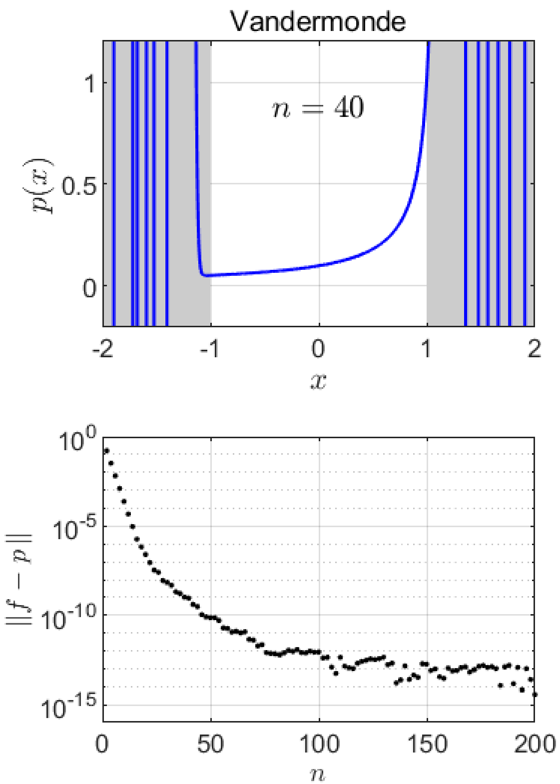

FIG. 5.1. *A Fourier extension example from [1], with $f(x) = 1/(10-9x)$ approximated over $[-1,1]$ by Fourier series scaled to the larger interval $[-2,2]$. This is equivalent to approximation by powers z^k over just half of the unit circle, leading to exponential ill-conditioning of the Vandermonde matrix.*

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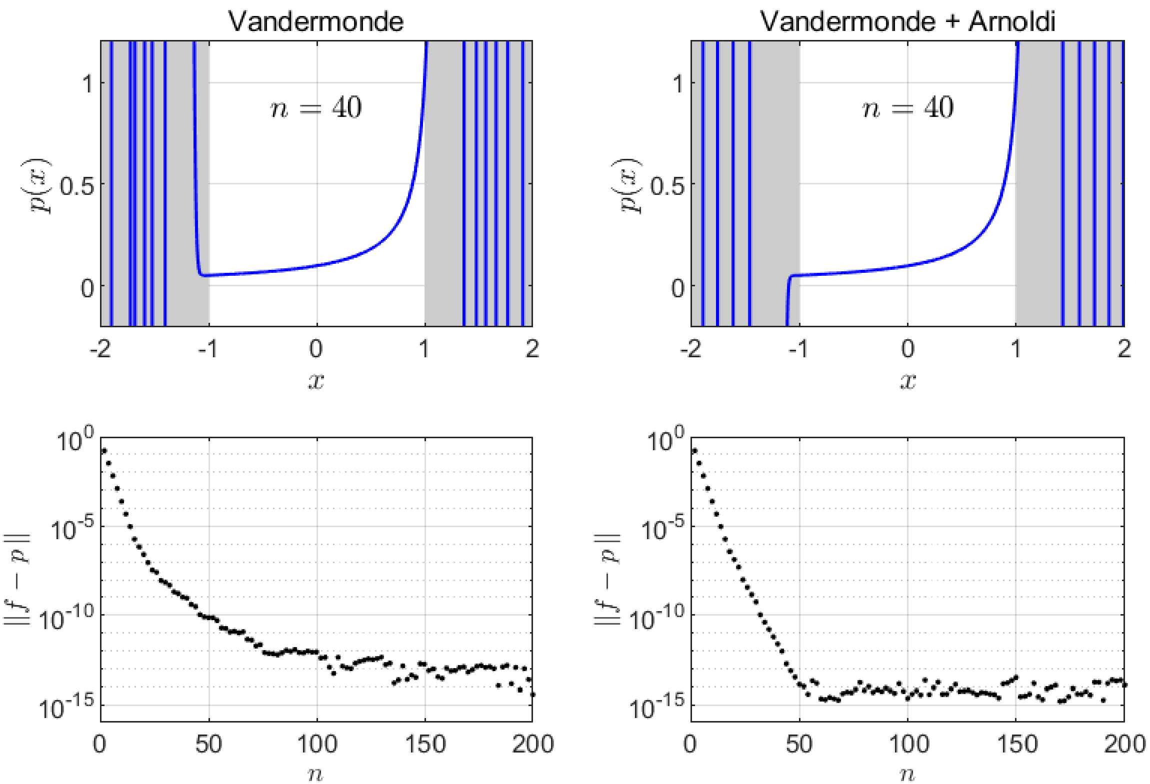

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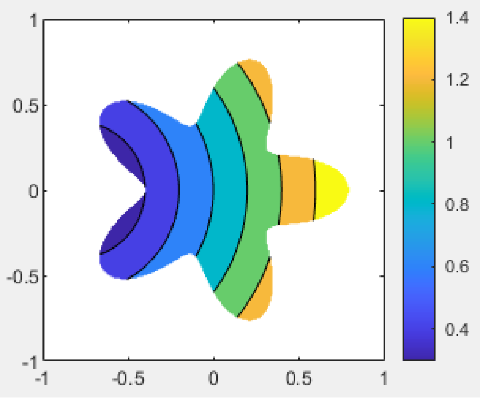
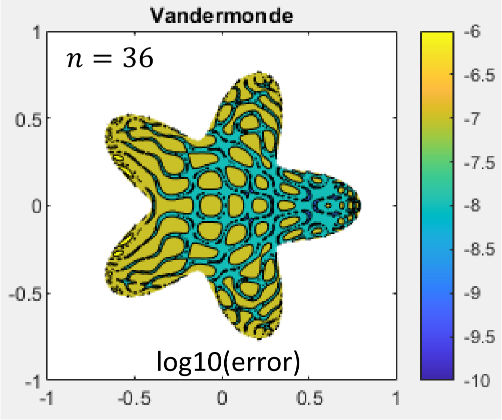
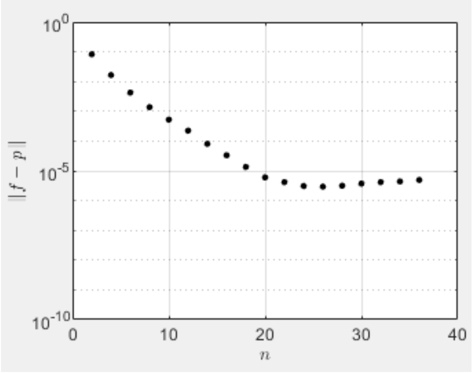
Example 5: Bivariate polynomial fit on a starfish domain

Hokanson, Nakatsukasa, T. + Webb, work in progress
See also Austin et al., arXiv, 2019.

It would be interesting to try Lawson iteration here too.







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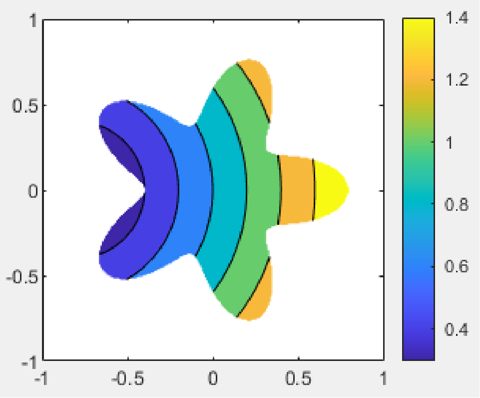

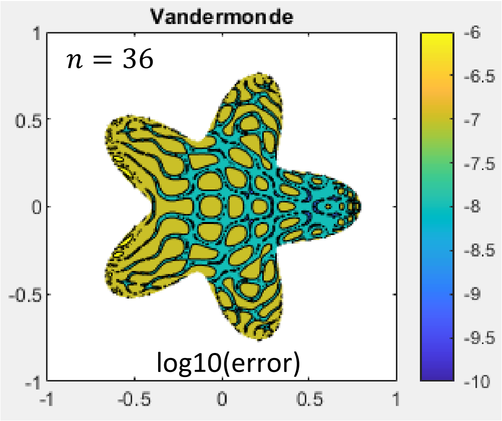

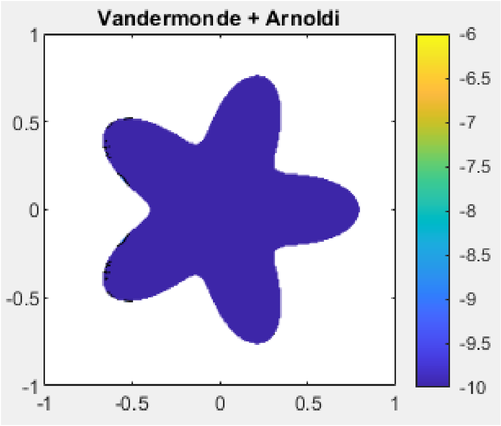

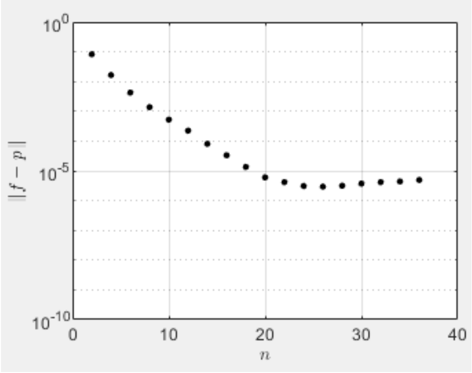

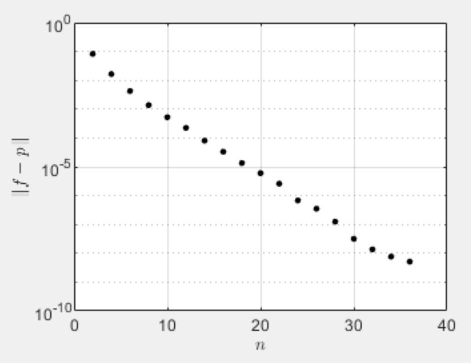

Example 6: Conformal mapping via polynomial approx of Green's function

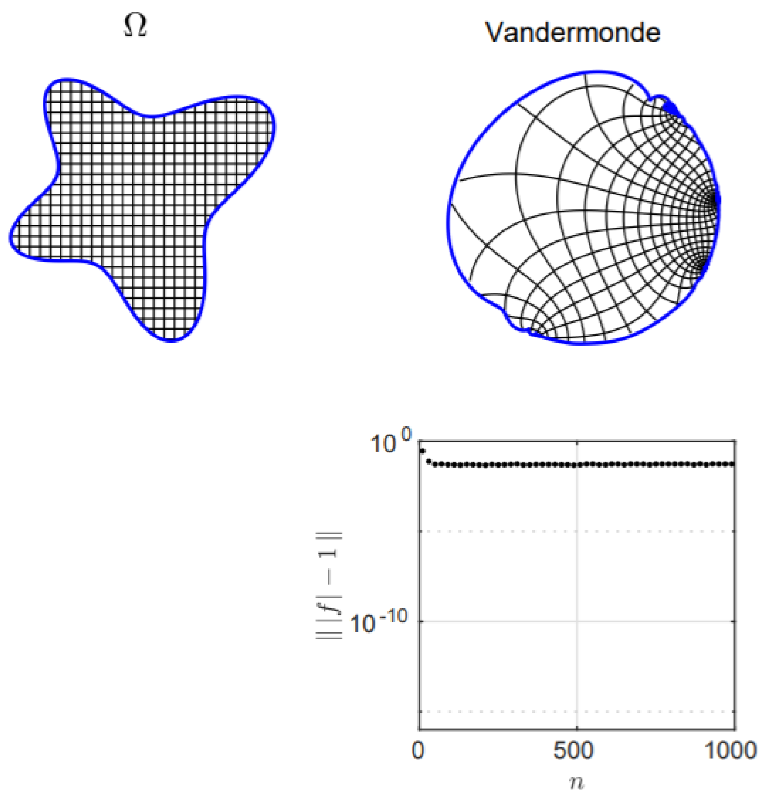

FIG. 6.1. *Conformal mapping of a blob onto the unit disk by the polynomial expansion method of* (6.1)–(6.4). *The two upper-right images correspond to* $n = 200$.

T., *Computational Methods and Function Theory*, to appear

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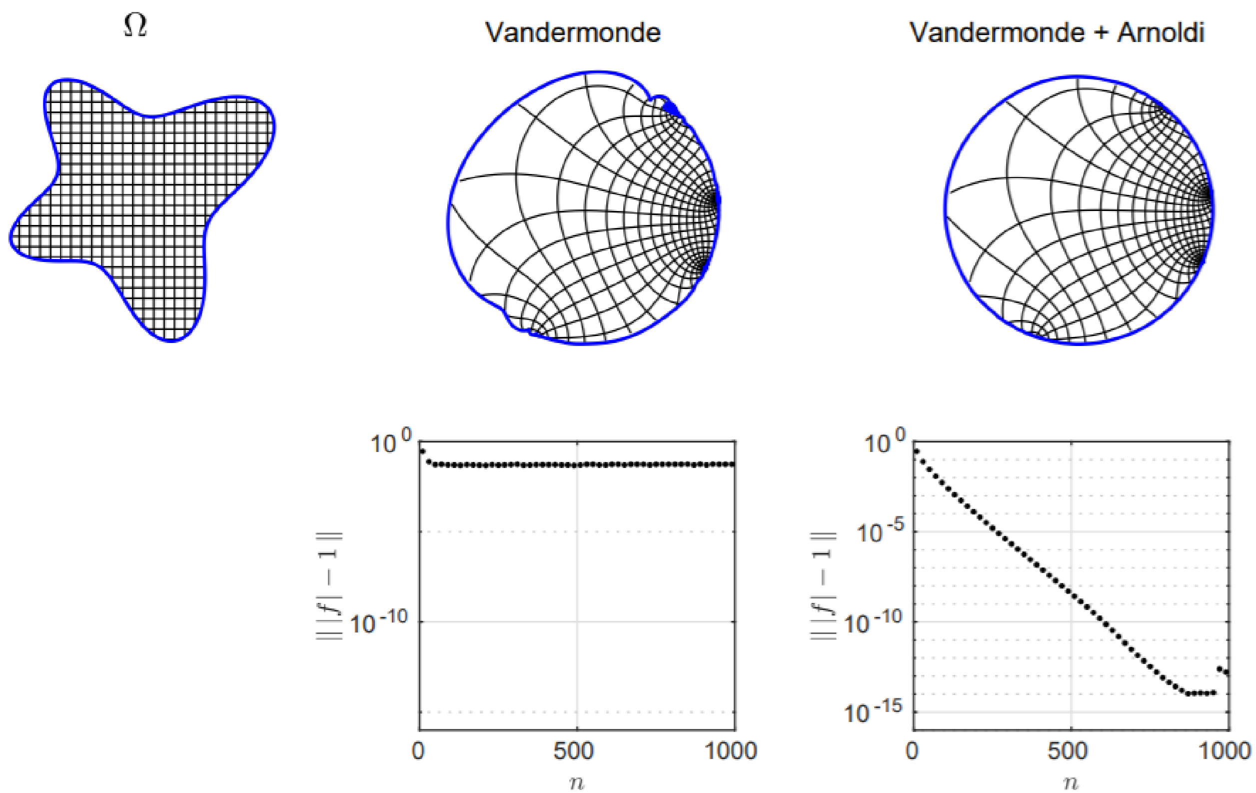

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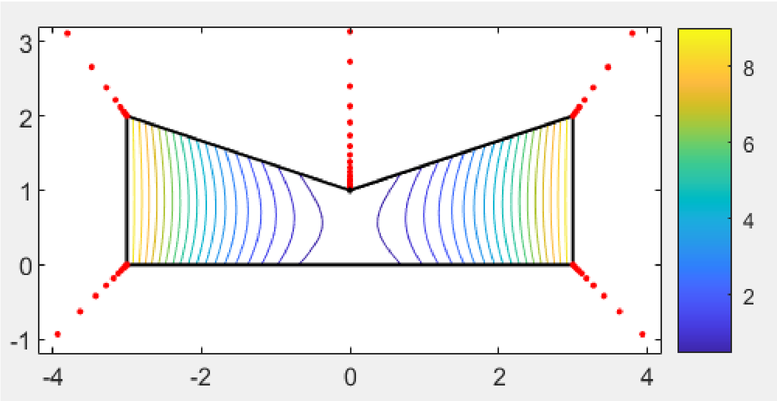
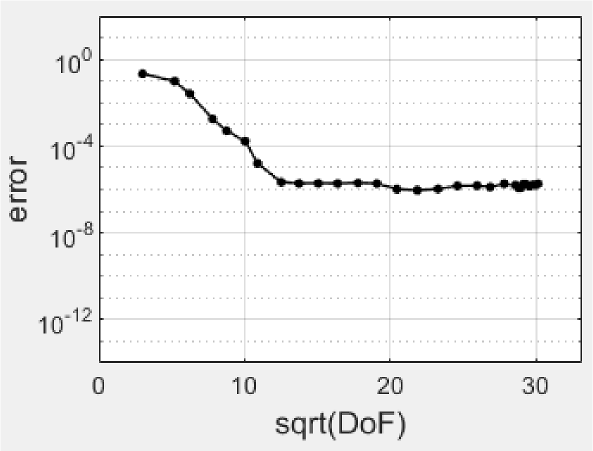
Example 7: Lightning Laplace solver

Gopal + T., *SINUM* 2019



Solution is approximated by real part of polynomial + rational function with exponentially clustered poles via least-squares on boundary.

without Arnoldi




```
P = [-3 3 3+2i 1i -3+2i];
laplace(P, 'tol', 1e-12, 'noarnoldi');
```

Demonstration of laplace.m and confmap.m
Codes available at Trefethen home page

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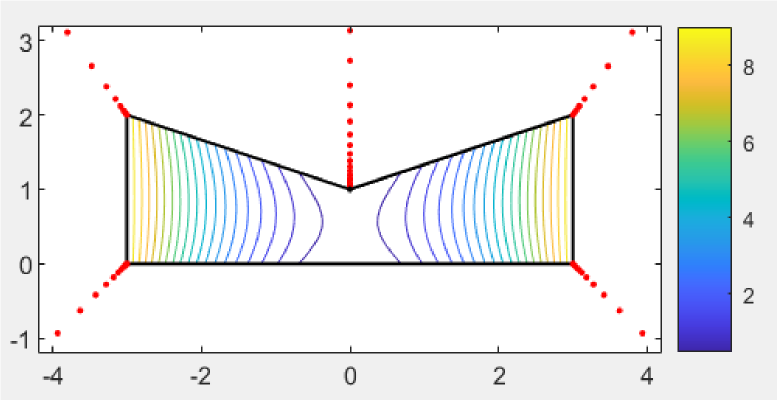

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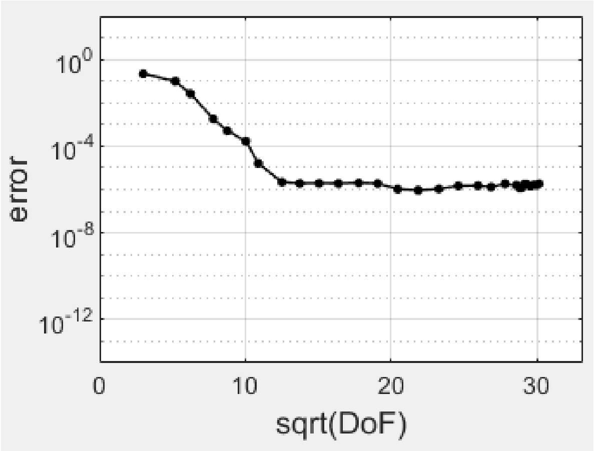

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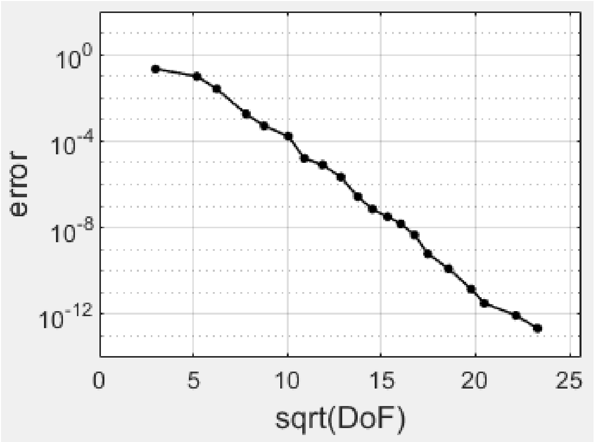


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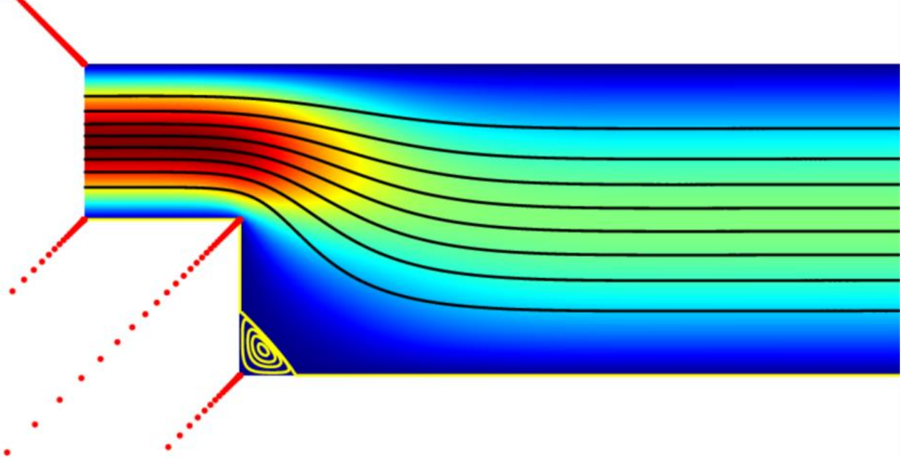
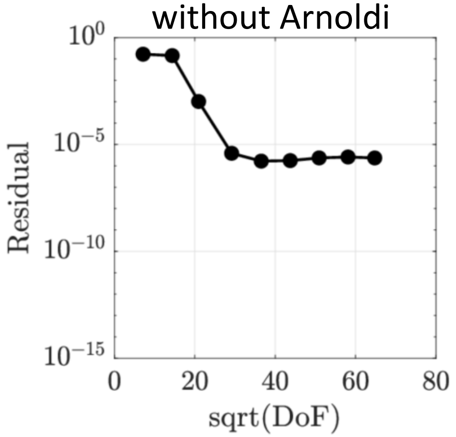
Example 8: Stokes flow

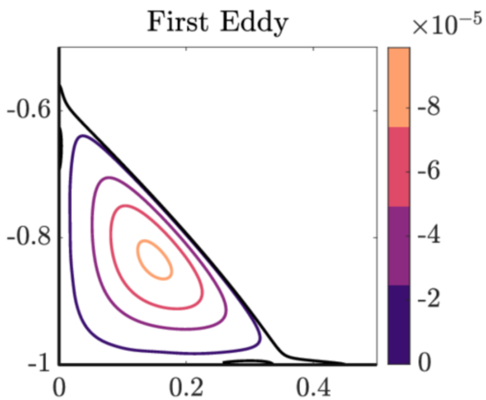


Brubeck + T., in preparation

Biharmonic equation is reduced to Laplace problem using Goursat representation.

without Arnoldi





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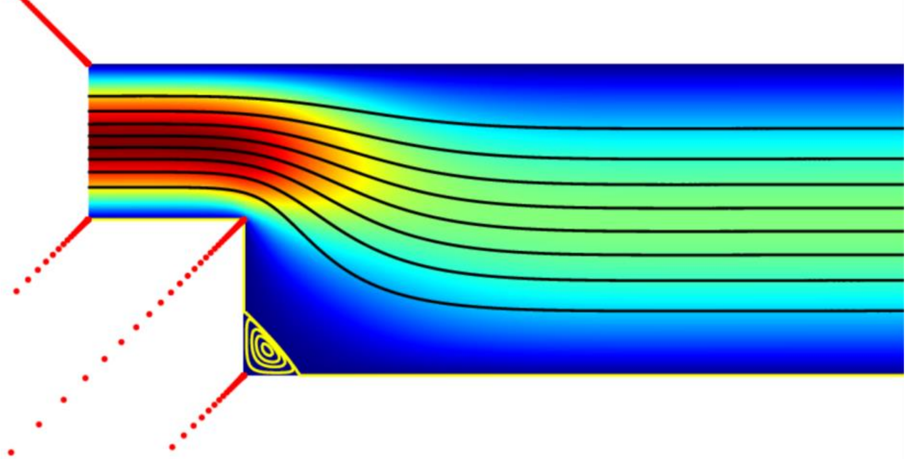

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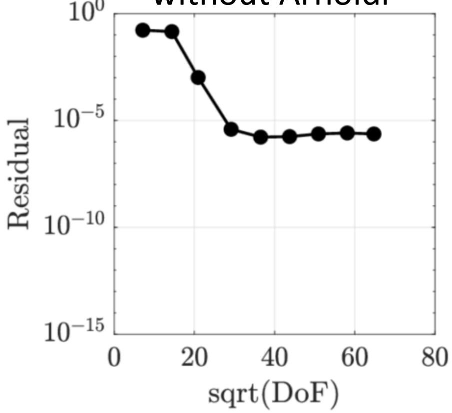

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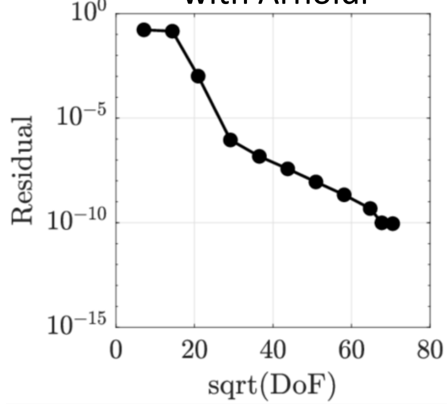

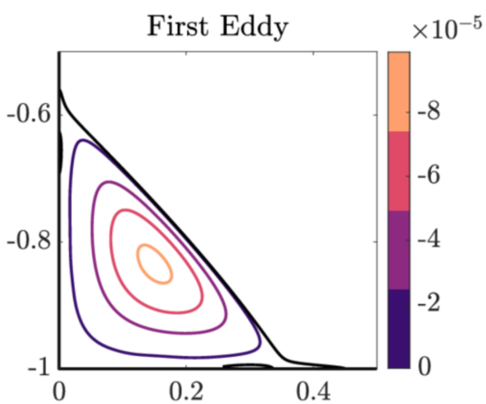

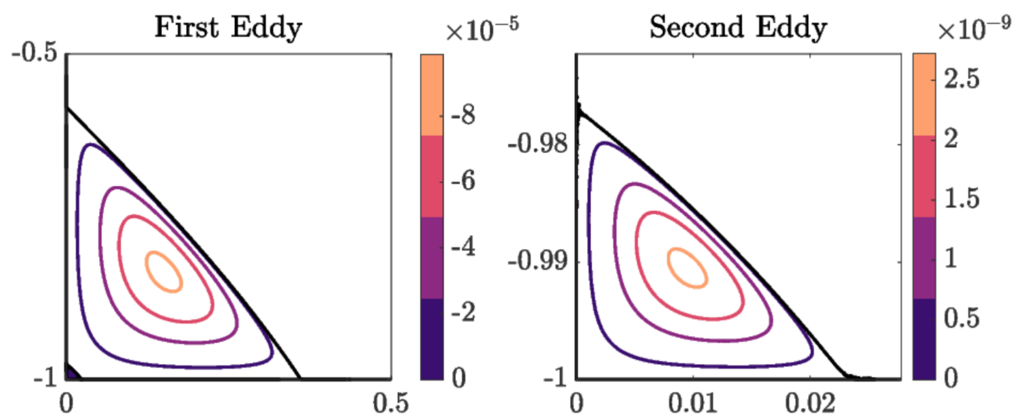

DISCUSSION

Numerical analysts tend to be expert at linear algebra
but relatively uninquisitive when it comes to basic issues of approximation.

For example we've seen this unfortunate message for decades:

```
>> x = 1:50;
>> y = -0.3*x + 2*randn(1,50);
>> p = polyfit(x,y,6);
Warning: Polynomial is badly conditioned. Add points with
distinct X values, reduce the degree of the polynomial, or try
centering and scaling as described in HELP POLYFIT.
> In polyfit (line 79)
```

In fact, this polynomial is not badly conditioned — only the basis $\{1, x, x^2, \ldots, x^6\}$.

A sociological and historical accident:

LINEAR ALGEBRA	APPROXIMATION
Dominated by numerical people	Dominated by theoretical people

Yet they are equally fundamental for numerical computation.