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AUTONOMOUS ROBOTS & ALGEBRAIC GEOMETRY

PREVENTING THE QUANTUM CRYPTO APOCALYPSE

NOTES OF A NUMERICAL ANALYST

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COVER IMAGE

Cover image: a pair of images of punts under Magdalen Bridge, Oxford, illustrating the theme of discrete and continuous in the *Notes of a Numerical Analyst* column (page 32). On the left, 32x48 pixels, and on the right, 512x768.

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Notes of a Numerical Analyst

Discrete and Continuous

NICK TREFETHEN FRS

There are many parallels between phenomena of linear algebra (discrete) and differential equations (continuous). Here are two of my favourites.

Sturm–Liouville. Consider Wilkinson's $(2n+1)\times(2n+1)$ tridiagonal matrix of the form

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \tag{1}$$

with the diagonal entries running from n down to 0 and up to n again. Theory tells us that the eigenvalues of A are distinct, though there's no such theorem for a pentadiagonal matrix. Yet the eigenvalues fall in nearly degenerate pairs, like these largest two for $n = 4$ and 8:

$$n = 4: \ 4.745, \ 4.747,$$

$$n = 8: \ 8.7461941826, \ 8.7461941832.$$

A continuous analogue is the Sturm–Liouville problem

$$y'' + |x|y = \lambda y, \quad -L < x < L$$

with $L > 0$ and $y(\pm L) = 0$. Again, theory tells us that the eigenvalues are distinct, though there's no such theorem for a fourth-order equation. Here are the largest eigenvalues for $L = 4$ and 8:

$$L = 4: \ 1.645, \ 1.682,$$

$$L = 8: \ 5.661892585, \ 5.661892595$$

These near-degeneracies are related to line- splitting effects in quantum mechanics.

Frozen coefficients. Suppose you have a family of matrices that are individually power-bounded, with all eigenvalues in the unit disk, like this pair:

$$A_1 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}.$$

Must products like $A_1A_2A_1A_2\cdots$ converge to zero? Certainly not, since A_1A_2 has eigenvalues 0 and 4.

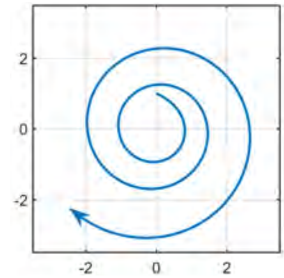

Figure 1. Solutions to a variable-coefficient linear ODE may diverge even though all the frozen-coefficient problems are stable.

A continuous analogue is the ODE

$$u' = Bu, \quad B = \begin{pmatrix} -1 & m \\ 0 & -1 \end{pmatrix},$$

which is stable since the eigenvalues are in the left half-plane. For $m > 2$ and $u(0) = [0,1]^T$, however, the solution $u(t)$ will grow before decaying. If we now define $A(t)$ to be B "rotated by t",

$$A(t) = S(t)BS(-t), \quad S(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix},$$

solutions to $u' = A(t)u$ grow exponentially, as shown in the figure (with $m = 2.2$). Such effects were investigated by Lyapunov, Poincaré, Perron, and Vinograd, and this example comes from a celebrated paper by Kreiss about discrete-continuous analogues [1]. This effect is related to why flow in a pipe becomes turbulent even though the linearized problem is stable.

FURTHER READING

[1] H.-O. Kreiss, Über die Stabilitätsdefinition für Differenzengleichungen die partielle Differenzialgleichungen approximieren, *BIT*, 2 (1962), 153–181.

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