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ON ℓ^p-INSTABILITY AND OSCILLATION AT DISCONTINUITIES IN FINITE DIFFERENCE SCHEMES

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Abstract

It is known that even-order finite difference models of hyperbolic partial differential equations generate spurious oscillations around discontinuities, and also that they are unstable in ℓ^p norms for $p\neq2$. This paper presents an elementary argument involving dispersion and dissipation which shows that these two phenomena are closely related, explains their physical basis, and reproduces the known estimates for the width of the region of oscillations and for the strength of the instability.

1. Introduction

It is well known that finite difference formulas for hyperbolic partial differential equations often suffer from spurious oscillations near discontinuities. As a model hyperbolic equation it is customary to consider the simple linear first-order wave equation

$$u_t=u_x, \quad u(x,0)=u_0(x), \qquad x\in(-\infty,\infty), \quad t\geq0. \tag{1}$$

Various precise results for the behavior of the region of oscillations generated by finite difference approximations to (1) have been obtained over the years by Apelkrans, Brenner, Chin, Hedstrom, Serdjukova, Thomée, and others [1,2,4,6,7]. In particular, Chin and Hedstrom [4] have shown by saddle-point analysis that the numerical solutions to (1) behave approximately like integrals of generalized Airy functions.

Another widely recognized fact, first proved by Thomée (1964, unpublished) and also mentioned on p. 100 of the book by Richtmyer and Morton [8], is that every finite difference model of (1) with even order of accuracy is unstable in the ℓ^p norm for every $p\in[1,\infty]$ with $p\neq2$. See also [5]. For example the leap frog, Lax-Wendroff, and Crank-Nicolson difference formulas are all ℓ^p-unstable. The instability is weak, but it may have undesirable consequences in extensions to nonlinear problems, where both ℓ^1 and ℓ^∞ have a natural significance. For this reason a number of mathematicians have studied ℓ^p-instability of finite difference formulas during the past twenty-five years, including Brenner, Hedstrom, Serdjukova, Stetter, Strang, Thomée, and Wahlbin. A wealth of results of this work are presented in the monograph [3]. Most of the proofs given there are based on techniques of Fourier multipliers, the specialization to constant coefficients of pseudodifferential operators.

The purpose of this brief paper is to show that both ℓ^p-instability and oscillations at discontinuities are caused by a single process of numerical dispersion. An even-order finite difference model is one for which dispersion dominates dissipation at low wave numbers, so that the wiggles introduced by dispersion are not all rapidly damped. By considering the highest wave number for which this remains true, and by estimating its associated group velocity, we will reproduce quantitatively the main results alluded to above. The argument is only heuristic, and indeed we state it very loosely so as not to obscure the physical idea with details. However, much of this can probably be made rigorous.

In two or more space dimensions, an easily visualized geometrical focusing process renders hyperbolic differential equations ill-posed in L^p. The reason that finite difference models are ℓ^p-unstable even in one space dimension is that the dispersion introduced by discretization can bring about a similar kind of focusing. This is analogous to the situation regarding (ℓ^2-) stability of initial boundary value problems: uncontrolled radiation of waves from the boundary can cause ill-posedness of a differential equation only in two or more dimensions, but it can cause instability of a finite difference model even in one dimension [10].

Acknowledgment. Gerald Hedstrom has contributed as much as anyone else to the topics discussed here, and shares my physical view of their underlying explanation. I am grateful to him for valuable advice and assistance on repeated occasions.

2. Estimates of dissipation and dispersion

Let Q be a consistent ℓ^2-stable finite difference model of (1) with constant real coefficients. The solution to Q is a function $v_j^n \approx u(jh,nk)$ defined on a grid with space step h and time step k=const×h. We will also write v(t) for $v^n \approx u(\cdot,nk)$. By substituting the wave $\exp(i(\xi x+\omega t))$ into Q, where ξ is the wave number and ω is the frequency, one obtains the numerical dispersion relation (in general complex) relating ξ and ω. Consistency implies that for ξ and ω near 0, this relation is a function with the expansion

$$\omega = \xi + A\xi(i\xi h)^{\alpha-1} + B\xi(i\xi h)^{\beta-1} + \dots, \tag{2}$$

where A and B are nonzero real constants, $\alpha\geq3$ is an odd integer called the order of dispersion, and $\beta\geq2$ is an even integer (possibly ∞) called the order of dissipation. The omitted terms are understood to have order greater than α if odd, greater than β if even. The order of accuracy of Q is $\min\{\alpha-1,\beta-1\}$, and is even if $\alpha<\beta$ (dispersion dominates dissipation at low wave numbers), or odd if $\alpha>\beta$ (dissipation dominates dispersion). An equivalent way to express (2) is to say that Q has the modified equation [12]

$$u_t = u_x + Ah^{\alpha-1}\frac{\partial^\alpha u}{\partial x^\alpha} + Bh^{\beta-1}\frac{\partial^\beta u}{\partial x^\beta}.$$

The idea here is that if Q is applied with smooth but nonconstant initial data, then the evolution of the solution in time will be described more accurately by this equation than by (1).

Let Q be applied with initial data containing a range of wave numbers centered at $\xi=0$. According to (2), the frequency ω corresponding to a real wave number $\xi\neq 0$ is not real unless $\beta=\infty$, but contains an imaginary component on the order of $\xi^\beta h^{\beta-1}$. (Here and throughout, we ignore constant factors and worry only about exponents.) It follows that as t increases, the energy in v(t) at this wave number will dissipate away on a time scale on the order of $\xi^{-\beta}h^{1-\beta}$. (We know the sign of B is such that they energy decays rather than grows, because Q was assumed to be ℓ^2-stable.) Conversely, at any fixed time t we can expect that energy at wave numbers with $\xi^{-\beta}h^{1-\beta}<<t$, i.e. $\xi>>t^{-1/\beta}h^{(1-\beta)/\beta}$, will have largely dissipated away, while energy with $\xi<<t^{-1/\beta}h^{(1-\beta)/\beta}$ will have decayed very little. In other words the wave numbers present to a significant degree in v(t) cover a range $\Delta\xi(t)$ of order of magnitude

$$\Delta\xi(t) \approx t^{-1/\beta}h^{(1-\beta)/\beta} \approx h^{-1}n^{-1/\beta}. \qquad (3)$$

The first expression here implies that if t is held constant and the mesh is refined, the range of wave numbers remaining at time t increases in proportion to the fractional power $h^{(1-\beta)/\beta}$. The second implies that if h is held constant and n increases, the range of wave numbers decreases like $n^{-1/\beta}$. Both of these interpretations are important, and so eqs. (4) and (5) below will also be written in two forms.

So much for dissipation. To quantify dispersion, we consider the range of group velocities present in v(t). Under the differential equation (1), all energy travels at velocity exactly -1. But under Q, the energy at each wave number travels instead approximately at the numerical group velocity $C(\xi) = -d\omega(\xi)/d\xi$ obtained by differentiating the numerical dispersion relation, provided that this quantity has negligible imaginary part. (For illustrations see [9,11].) From here on let us assume $\alpha<\beta$, so that Q is of even order. Then by (2), the group velocity is

$$C(\xi) = -1 + O((\xi h)^{\alpha-1}),$$

and its imaginary part is indeed negligible so long as ξ is not too large. From (3) it follows that the wave components present in v(t) span a range of group velocities $\Delta C(t)$ of the order

$$\Delta C(t) \approx t^{(1-\alpha)/\beta}h^{(\alpha-1)/\beta} \approx n^{(1-\alpha)/\beta}. \qquad (4)$$

3. Oscillations around a discontinuity

Suppose the initial distribution contains a discontinuity of some kind. The ensuing oscillations result from the fact that the various sinusoidal components composing this signal travel at different group velocities under Q, and therefore separate with time. At time t the width $\Delta x(t)$ of the train of oscillations will be on the order of $t\Delta C(t)$, or by (4),

WIDTH OF REGION OF OSCILLATIONS

$$\Delta x(t) \approx t^{(\beta+1-\alpha)/\beta}h^{(\alpha-1)/\beta} \approx hn^{(\beta+1-\alpha)/\beta}. \qquad (5)$$

From the second estimate we see that the number of grid points covered by spurious oscillations grows in proportion to a fractional power of n as $n\to\infty$, if $\beta<\infty$. In the nondissipative case $\beta=\infty$ we get the limiting result of linear growth $\Delta x(t) \approx hn$.

Equation (5) matches the results presented in various forms in [1,2,4,6,7].

4. ℓ^p-instability

The same dispersion process just described also explains why Q is ℓ^p-unstable for $1\le p<2$. Given a number $t>0$, construct initial data for Q consisting of a narrow spike composed of wave numbers in the same range $\pm\Delta\xi$ as in (3),

$$\Delta\xi(0) \approx h^{-1}n^{-1/\beta}. \qquad (6)$$

The point of this choice of wave numbers is that by the reasoning of Sec. 2, the energy will not dissipate significantly up to time t: (3) will be valid, hence also (4) and (5). On the other hand since the range of wave numbers is limited, the initial spike can have width no smaller than $1/\Delta\xi$, i.e.

$$\Delta x(0) \approx hn^{1/\beta}. \qquad (7)$$

From (5) and (7) we see that the initial spike increases in width up to time t by a ratio

$$\frac{\Delta x(t)}{\Delta x(0)} \approx n^{(\beta-\alpha)/\beta}. \qquad (8)$$

Since the signal broadens, it must lose amplitude. Assume that at each time, v is a wave packet of more or less regular shape with amplitude $\|v\|_\infty$. (To achieve this, some care in construction of the initial spike will actually be necessary.) The ℓ^p norm $\|v\|_p$ can then be expected to have order of magnitude

$$\|v\|_p \approx (\Delta x)^{1/p}\|v\|_\infty \approx (\Delta x)^{\left(\frac{1}{p}-\frac{1}{2}\right)}\|v\|_2 .$$

By the construction of (6), $\|v\|_2$ will not change much from time 0 to t. From (8) we therefore conclude

$$\frac{\|v(t)\|_p}{\|v(0)\|_p} \approx \left(\frac{\Delta x(t)}{\Delta x(0)}\right)^{\left(\frac{1}{p}-\frac{1}{2}\right)} \approx n^{\frac{\beta-\alpha}{\beta}\left(\frac{1}{p}-\frac{1}{2}\right)} \qquad (9)$$

For $p<2$ the exponent in (9) is positive, so this construction generates growth in the ℓ^p norm which gets worse as n increases, even if h and k are decreased with t held constant. This implies that Q is unstable in ℓ^p, and in fact that the powers of the associated discrete solution operator S grow at least as fast as in (9),

$$\|S^n\|_p \gtrsim n^{\frac{\beta-\alpha}{\beta}\left(\frac{1}{p}-\frac{1}{2}\right)}.$$

We have shown that ℓ^p-instability for $p<2$ can be explained by the dispersion of a narrow spike into an oscillatory wave train. Conversely, instability for $p>2$ is due to the fact than an oscillatory wave train may coalesce into a spike (see Fig. 12 of [9]). To make this happen, in effect we need to reverse time in the above example. Equivalently, let v_j^n from the previous experiment be taken as new initial data v_j^0 for Q. Now if n additional time steps are taken, the initially broad wave packet will gather into a narrow spike again. For this process the estimates (8) and (9) are simply inverted, giving

$$\frac{\|v(t)\|_p}{\|v(0)\|_p} \approx n^{\frac{\beta-\alpha}{\beta}\left(\frac{1}{2}-\frac{1}{p}\right)}, \qquad (10)$$

that is, growth in ℓ^p for $p>2$ and decay for $p<2$. Combining (9) and (10) now gives the general growth rate bound

INSTABILITY IN ℓ^p

$$\|S^n\|_p \gtrsim n^{\frac{\beta-\alpha}{\beta}\left|\frac{1}{2}-\frac{1}{p}\right|}. \qquad (11)$$

Thus Q is unstable in ℓ^p for all $p\neq 2$, as we set out to show. Of course, the instability is very weak. For example, $\|S^n\|_1$ and $\|S^n\|_\infty$ grow at the rates $n^{1/2}$ and $n^{1/8}$ for leap frog and Lax-Wendroff, respectively.

The result (11) is exactly the growth rate obtained rigorously by Brenner, et al. by techniques of Fourier multipliers. In fact Theorems 5.3.1 and 5.3.2 of [2] establish a two-sided bound

$$M_1 n^{\frac{\beta-\alpha}{\beta}\left|\frac{1}{2}-\frac{1}{p}\right|} \le \|S^n\|_p \le M_2 n^{\frac{\beta-\alpha}{\beta}\left|\frac{1}{2}-\frac{1}{p}\right|} \qquad (12)$$

for some constants M_1, M_2, under suitable assumptions. (If $\beta=0$, each term $(\beta-\alpha)/\beta$ gets replaced by 1.) Thus our dispersion argument accounts for the full extent of instability of finite difference models in ℓ^p. By making the argument rigorous one could presumably reproduce the lower bound (first inequality) in (12). The upper bound must be obtained by other means.

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