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The Asymptotic Accuracy of Rational Best Approximations to e^z on a Disk

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The method described by D. Braess (*J. Approx. Theory* **40** (1984), 375–379) is applied to study approximation of e^z on a disk rather than an interval. Let E_{mn} be the distance in the supremum norm on $|z| \leqslant \rho$ from e^z to the set of rational functions of type (m, n). The analog of Braess' result turns out to be

$$E_{mn} \sim \frac{m!\, n!\, \rho^{m+n+1}}{(m+n)!\,(m+n+1)!} \qquad \text{as} \quad m+n \to \infty.$$

This formula was obtained originally for a special case by E. Saff (*J. Approx. Theory* **9** (1973), 97–101).

In this paper we apply the origin-shift idea of the preceding paper by Braess [1] to obtain the corresponding result for approximation of e^z on a disk in the complex plane. Let $m, n \geqslant 0$ be integers, and let R_{mn} be the set of rational functions of type (m, n). Let E_{mn} denote the error in best Chebyshev approximation of type (m, n) to e^z on the disk $|z| \leqslant \rho$ for some $\rho \geqslant 0$, i.e.,

$$E_{mn} = \inf_{r \in R_{mn}} \|e^z - r\|,$$

where $\|\phi\| = \sup_{|z| \leqslant \rho} |\phi(z)|$. We will show

THEOREM 1.

$$E_{mn} = \frac{m!\, n!\, \rho^{m+n+1}}{(m+n)!\,(m+n+1)!}(1 + o(1)) \qquad \text{as} \quad m+n \to \infty. \tag{1}$$

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This formula has been previously shown valid by E. Saff [2] for the special case in which n is fixed as $m \to \infty$, i.e., for approximation along rows in the Walsh table.

The basis of Braess' ingenious proof is to make use of a Padé approximant to e^z not at the point 0, but at $z_0 = (m+3n)/4(m+n)(m+n+1)$. Our modification for the disk $|z| \leqslant \rho$ uses a Padé approximant at $z_0 = 2n\rho^2/(m+n)(m+n+1)$. The error curves for these shifted approximants approach circles as $m+n \to \infty$, which implies that their errors approach the optimal values E_{mn} [3, 4].

Comparing (1) with Braess' Eq. (1), one sees that one pays a price of a factor 2^{m+n} in expanding the domain of approximation from the unit interval to the unit disk.

Proof of Theorem 1. Let $r = p/q \in R_{mn}$ be the Padé approximant of type (m, n) to e^z (at the origin), normalized as in Braess' paper by $p(0) = q(0) = (m+n)!$. Following Perron, Braess obtains the formula

$$e^z q(z) - p(z) = (-1)^n z^{m+n+1} \int_0^1 (1-u)^m u^n e^{uz}\, du. \tag{2}$$

As $m+n \to \infty$, the integrand here becomes an increasingly narrow spike centered at $u = n/(m+n)$. If the term e^{uz} were not present, the value of the integral would be $m!\, n!/(m+n+1)!$. Therefore with that term, it will be

$$\int_0^1 (1-u)^m u^n e^{uz}\, du = \frac{m!\, n!}{(m+n+1)!} e^{(n/(m+n))z}(1+o(1)) \qquad \text{as} \quad m+n \to \infty$$

uniformly for z in any compact subset of the plane. Inserting this result in (2) gives

$$e^z q(z) - p(z) = \frac{m!\, n!\, (-1)^n z^{m+n+1}}{(m+n+1)!} e^{(n/(m+n))z}(1+o(1)) \qquad \text{as} \quad m+n \to \infty. \tag{3}$$

(Here we have simplified the argument leading to Braess' Eq. (6), to make it clear that (3) holds on any compact set, not just for $|z| \leqslant \frac{1}{2}$.) Dividing this by Braess' Eq. (7), we obtain

$$e^z - r(z) = \frac{m!\, n!\, (-1)^n}{(m+n)!\, (m+n+1)!} e^{(2n/(m+n))z} z^{m+n+1}(1+o(1)) \qquad \text{as} \quad m+n \to \infty \tag{4}$$

uniformly for z in any compact set. This formula implies that the error curve for r, i.e., the image of $|z| = \rho$ under $e^z - r(z)$, is asymptotically an

$(m+n+1)$-winding curve that varies in modulus by a factor $e^{4n\rho/(m+n)} \leqslant e^{4\rho}$.

Now let $\tilde{r} \in R_{mn}$ be the (m, n) Padé approximant to e^z at the point

$$z_0 = \frac{2n\rho^2}{(m+n)(m+n+1)}, \tag{5}$$

that is, $\tilde{r}(z) = e^{z_0} r(z - z_0)$. Then by (4) we have

$$\begin{aligned} e^z - \tilde{r}(z) &= e^{z_0}(e^{z-z_0} - r(z-z_0)) \\ &= \frac{m!\, n!\, (-1)^n e^{z_0}}{(m+n)!\, (m+n+1)!}\, e^{(2n/(m+n))(z-z_0)} (z-z_0)^{m+n+1} (1+o(1)) \\ &= \frac{m!\, n!\, (-1)^n}{(m+n)!\, (m+n+1)!}\, e^{(2n/(m+n))z} (z-z_0)^{m+n+1} (1+o(1)) \\ &\quad \text{as} \quad m+n \to \infty \end{aligned}$$

since $z_0 \to 0$. To show that this function maps $|z| = \rho$ onto a curve that approaches a circle of radius $m!\, n!\, \rho^{m+n+1}/(m+n)!\, (m+n+1)!$ as $m+n \to \infty$, and thereby prove (1) (see Prop. 2.2 of [4]), we need to show

$$|z - z_0|^{m+n+1} = \rho^{m+n+1}\, |e^{(-2n/(m+n))z}|\, (1+o(1)) \qquad \text{as} \quad m+n \to \infty$$

for $|z| = \rho$. Of course z_0 was chosen to make this happen. We compute for $|z| = \rho$

$$\begin{aligned} |z - z_0|^{m+n+1} = \rho^{m+n+1}\, |1 - z_0/z|^{m+n+1} &= \rho^{m+n+1} \left| 1 - \frac{z_0 z}{\rho^2} \right|^{m+n+1} \\ &= \rho^{m+n+1} \left| 1 - \frac{2nz}{(m+n)(m+n+1)} \right|^{m+n+1} \\ &= \rho^{m+n+1}\, |e^{(-2n/m+n))z}|\, (1+o(1)), \end{aligned}$$

as required. ∎

Note that since $z_0 = 0$ for $n = 0$, no origin shift is needed to establish (1) in the first row of the Walsh table. In fact the same is true in any row (i.e., for any fixed n), and this is the basis of Saff's proof mentioned above [2].

The estimate (4) is interesting in its own right, for it describes the degree of optimality of (standard) Padé approximants to e^z, and also the circularity of their error curves. It appears that these results have not been recorded before:

THEOREM 2. *Let $E_{\min}$ and $E_{\max}$ denote the minimum and maximum errors $|e^z - r(z)|$ for $|z| = \rho > 0$, where $r \in R_{mn}$ is the (m, n) Padé approximant to e^z. If m and n increase to ∞ along a ray at angle $\theta \in [0, \pi/2]$ from the m-axis, i.e., $m + n \to \infty$ with* $\arctan(n/m) \to \theta$, *then*

$$\text{(a)} \quad \frac{E_{\max}}{E_{mn}} \to \exp\left(\frac{2\rho}{1 + \cot\theta}\right) \qquad \textit{as} \quad m + n \to \infty \tag{6}$$

and

$$\text{(b)} \quad \frac{E_{\min}}{E_{\max}} \to \exp\left(\frac{-4\rho}{1 + \cot\theta}\right) \qquad \textit{as} \quad m + n \to \infty. \tag{7}$$

Proof. The second result follows from (4), and the first from (4) and (1). ∎

It is worth mentioning that the error curves for true best approximations to e^z are much more nearly circular than those for the shifted Padé approximants employed in the proof of Theorem 1. For example, in the case $(m, n) = (1, 1)$ and $\rho = 1$ the standard Padé approximation has $E_{\min}/E_{\max} \approx 0.123$ (Fig. 1 of [4]), which is not far from the estimate $e^{-2} \approx 0.135$ of (7). When the origin is shifted to $z_0 = \frac{1}{3}$ following (5), this ratio increases to 0.813, and if it is shifted to the point $z_0' \approx 0.306$ that gives the most nearly circular error curve, it increases further to 0.958. But all of these numbers are far from the value 0.993 for the true best approximation (Table 1 of [4]). In best approximation of analytic functions other than e^z it is quite typical for error curves to approach circles as $m + n \to \infty$ [3, 4], but there is no reason to expect that such behavior can often be achieved by shifted Padé approximants.

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