Wide-angle one-way wave equations

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A one-way wave equation, also known as a paraxial or parabolic wave equation, is a differential equation that permits wave propagation in certain directions only. Such equations are used regularly in underwater acoustics, in geophysics, and as energy-absorbing numerical boundary conditions. The design of a one-way wave equation is connected with the approximation of $(1-s^2)^{1/2}$ on $[-1,1]$ by a rational function, which has usually been carried out by Padé approximation. This article presents coefficients for L^2, L^∞, and other alternative classes of approximants that have better wide-angle behavior. For theoretical results establishing the well posedness of these wide-angle equations, see the work of Trefethen and Halpern ["Well-posedness of one-way wave equations and absorbing boundary conditions," Math. Comput. **47**, 421–435 (1986)].

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INTRODUCTION

The constant-coefficient wave equation

$$u_{tt} = u_{xx} + u_{yy} \tag{1}$$

admits plane-wave solutions traveling in all directions. However, in several applications it is desirable to replace (1) by a *one-way wave equation* (OWWE), an equation that permits wave propagation in a 180° range of angles only. In recent years this idea has become a standard tool in underwater acoustics,[1,2] in geophysics,[3–5] and in the design of numerical "absorbing boundary conditions."[6–8]

The purpose of this article is to present coefficients for a number of families of wide-angle one-way equations. A "wide-angle" one-way equation is one designed to be accurate over nearly the whole 180° range of permitted angles, not just a small subrange; this idea has been exploited by Greene,[9] for example, who proposed equations similar to the "L^∞_α" approximations in Secs. II and III below. As one-way wave equations become better understood and computers grow more powerful, wide-angle accuracy is becoming an issue of greater importance. The construction of such formulas has usually been carried out on an *ad hoc* basis, but we will aim to be more systematic here by drawing upon the connection with the mathematical field of approximation theory. Some related results based on interpolation can be found in the recent papers of Higdon.[10,11]

The theorems in our previous paper[12] establish that all of the one-way wave equations considered here are well posed.

To make our results broadly applicable, we have intentionally confined our attention to the very simplest equation (1), and omitted all details of physics and of numerical implementation (both are problem dependent). We are well aware that the constant coefficient in (1) is unrealistic for many problems, including those of underwater acoustics. Some one-way wave problems also involve fundamentally different mathematics, such as the Navier–Stokes equations, the small-disturbance equations of transonic flow, the shallow-water equations, and the equations of elasticity. Another variation of (1) is that in many applications the equation is reduced by assuming a fixed frequency ω, so that t drops out; fortunately, this can be easily accomplished either before or after the analysis presented here. We ask the reader to bear with these oversimplifications in the interest of being able to consider a wide range of approximations in a short space.

Our purpose is to describe some candidates, not pick a favorite. A responsible judgment of the merits of these approximations must be based on extensive computations, and the conclusions will depend on the field of application. Some initial comparative studies of this kind have been carried out recently by St. Mary and Lee[13] and by Blaschak and Kriegsmann.[14]

I. ONE-WAY WAVE EQUATIONS AND RATIONAL APPROXIMATION OF $\sqrt{1-s^2}$

If one substitutes the plane wave

$$u(x,y,t) = e^{i(\omega t + \xi x + \eta y)} \tag{2}$$

into (1), where ω is the frequency and ξ and η are the x and y wavenumbers, the result is the *dispersion relation*

$$\omega^2 = \xi^2 + \eta^2, \tag{3}$$

or equivalently

$$\xi = \pm\omega\sqrt{1-\eta^2/\omega^2}\,. \tag{4}$$

This is the equation of a circle in the $\xi/\omega - \eta/\omega$ plane corresponding to plane waves propagating in all directions. The wave with wavenumbers ξ, η has velocity $(-\xi/\omega, -\eta/\omega) = (-\cos\theta, -\sin\theta)$, where θ is the angle counterclockwise from the negative x axis. By taking the plus or minus sign in (4) only, however, we can restrict attention to leftgoing ($|\theta| \leqslant 90°$) or rightgoing ($|\theta| \geqslant 90°$) waves, respectively. For algebraic simplicity, we will choose the former course and write

$$\xi = +\omega\sqrt{1-s^2},$$

DISPERSION RELATION FOR IDEAL OWWE, (5)

with

$$s=\eta/\omega=\sin\theta\in[-1,1], \quad \theta\in[-90°,90°]. \qquad (6)$$

See Fig. 1.

Because of the square root, (5) is not the dispersion relation of any partial differential equation, but of a pseudodifferential equation. The idea behind practical one-way wave equations is to replace the square root by a rational function $r(s)$ of type (m,n) for some m and n; that is, the ratio of a polynomial p_m of degree m and a polynomial q_n of degree n:

$$r(s)=p_m(s)/q_n(s).$$

Then (5) becomes

$$\xi=\omega r(s),$$

DISPERSION RELATION FOR APPROXIMATE OWWE, (7)

for the same range (6) of s and θ. By clearing denominators, we can transform (7) into a polynomial of degree $\max\{m,n+1\}$ in ω, ξ, and η, and this is the dispersion relation of a true differential equation. For example, suppose $r(s)$ is the type (2,2) Padé approximant,

$$r(s)=(1-\tfrac{3}{4}s^2)/(1-\tfrac{1}{4}s^2),$$

which interpolates $\sqrt{1-s^2}$ six times at the origin, i.e., $r(s)-\sqrt{1-s^2}=O(s^6)$. Then (7) becomes

$$\xi\left(1-\frac{1}{4}\frac{\eta^2}{\omega^2}\right)=\omega\left(1-\frac{3}{4}\frac{\eta^2}{\omega^2}\right)$$

or

$$\xi\omega^2-\tfrac{1}{4}\xi\eta^2=\omega^3-\tfrac{3}{4}\omega\eta^2,$$

which corresponds to the differential equation

$$u_{xtt}-\tfrac{1}{4}u_{xyy}=u_{ttt}-\tfrac{3}{4}u_{tyy}. \qquad (8)$$

This is sometimes known as the 45° equation, because it has high accuracy approximately in the range $|\theta|\leqslant 45°$.

Thus at an abstract level, designing one-way wave equations can be reduced to the problem of finding rational approximations $r(s)$ to $\sqrt{1-s^2}$ for $s\in[-1,1]$, i.e., for $\theta\in[-90°,90°]$. (In the literature, the derivation is often not presented this way, but the approximation of the square root is implicit.) Figure 2 illustrates this approximation problem. For simplicity, we will restrict attention to functions $r(s)$ that are even [i.e., $r(-s)=r(s)$], which implies that m and n are even (see Ref. 12 for the general case). The question is, what approximation strategies will lead to the best results?

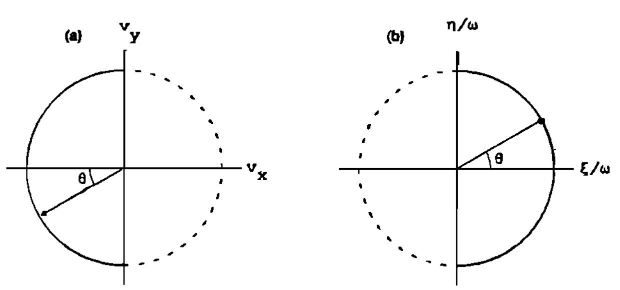


FIG. 1. Notation for the one-way wave equation.

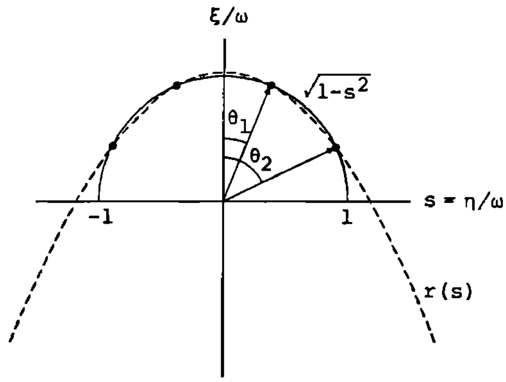


FIG. 2. The approximation problem: $r(s)\approx\sqrt{1-s^2}$.

As mentioned above, the standard choice in the past has been Padé approximation, which refers to maximal-order interpolation at the origin $s=0$, $\theta=0°$. (Padé approximation is equivalent to the truncation of an appropriate continued fraction expansion.) For a low order such as $(m,n)=(0,0)$, not much choice is available, but as one moves to approximations of type (2,0), (2,2), or higher, it becomes less clear that it is a good idea to concentrate all of the points of interpolation in one place. In some applications, it may be better to spread them throughout $s\in[-1,1]$, $\theta\in[-90°,90°]$. That is the idea behind the alternative families of approximations in this article.

The most accurate approximants are generally obtained with $m=n$ or $m=n+2$, and in some applications, these are also the only ones that lead to well-posed differential equations. Therefore, we will restrict our attention to these two cases. Given m and n, it is possible to find a rational function $r(s)$ that interpolates $\sqrt{1-s^2}$ in an arbitrary set of $m+n+2$ points symmetrically located in $[-1,1]$, but no more than this. We will assume that the number of points of interpolation is exactly $m+n+2$, which guarantees well posedness. (See Ref. 12 regarding all of the above assertions.) Moreover, as Higdon has pointed out,[10] an approximation with fewer than $m+n+2$ points of interpolation can always be improved at every point $s\in[-1,1]$ and hence is of no practical interest. By the formula given above, the order of the corresponding one-way wave equation will be $K=\tfrac{1}{2}(m+n+2)$. Let $\pm s_1,\ldots,\pm s_K$ denote the $2K$ points of interpolation in $[-1,1]$, counted with multiplicity, and $\pm\theta_1,\ldots,\pm\theta_K$ the corresponding angles in $[-90°,90°]$. These are the angles at which the one-way approximation will be exact. Here is a summary of the notation:

Rational type of $r(s)$:

$$(m,n), \quad m,n=0,2,4,\ldots, \quad m=n \text{ or } m=n+2.$$

Order of one-way wave equation:

$$K=\tfrac{1}{2}(m+n+2)=1,2,3,\ldots.$$

Number of points of interpolation in $[-1,1]$:

$$2K=2,4,6,\ldots.$$

Points of interpolation:

$$\pm s_1=\pm\sin\theta_1,\ldots,\pm s_K=\pm\sin\theta_K.$$

The interpolating polynomial can be constructed by a mechanical procedure which goes back at least to Newman in 1964.[15] Let p be a nonzero polynomial of degree K that is zero at $\sqrt{1-s_k^2}$ for each k, and set

$$r(s) = \frac{p(t) + p(-t)}{[-p(t) + p(-t)]/t}, \tag{9}$$

where $t = \sqrt{1-s^2}$. Since the numerator is even as a function of t, it is a polynomial in s of degree m, and since the denominator is even as a function of t, it is a polynomial in s of degree n. Also, since $|p(-t)| > |p(t)|$ for $t>0$ and $p'(0) \neq 0$, $r(s)$ can have no poles or zeros in $[-1,1]$. [We assume that at most one s_k is 1, and we exclude the trivial case $K=1$, $s_k = 1$, $r(s) \equiv 0$.] Thus $r(s) = \sqrt{1-s^2}$ is equivalent to

$$p(t) + p(-t) = -p(t) + p(-t),$$

that is, $p(t) = 0$. In other words, (9) interpolates $\sqrt{1-s^2}$ at the points $\pm s_k$.

II. SEVEN FAMILIES OF APPROXIMATIONS

This section defines the seven families of approximations we have chosen to consider, and indicates how their coefficients can be computed. For families 1, 2, 6, 7, the approximations are known essentially in closed form, while families 3, 4, 5 require iterative computations. The next section gives numerical results.

A. Padé

In Padé approximation, all $m+n+2$ points of interpolation coincide at $s = \theta = 0$; when $n = 0$ the Padé approximant is a truncated Taylor series. The type (0,0) Padé one-way wave equation is $u_x = u_t$, the "5° equation," which is inadequate for most purposes. The type (2,0) Padé one-way wave equation is $u_{xt} = u_{tt} - \frac{1}{2} u_{yy}$, the "15° equation" or the original "parabolic wave equation," and this has been used in numerous applications. The type (2,2) Padé one-way wave equation given in (8), the "40°" or "45° equation," has also been used fairly often, for example by Claerbout and Clayton in geophysics,[4,5] by Engquist and Majda for absorbing boundary conditions,[6,7] and by Botseas *et al.* in underwater acoustics.[16] The (4,2) and (4,4) approximants have been proposed in Ref. 17 and Refs. 18 and 19, respectively.

Padé approximants to $\sqrt{1-s^2}$ can be computed by the general interpolation procedure of the last section, summarized in (9), and this is what we have done in our computer program. An alternative approach is to use the fact that the Padé approximants of types (0,0), (2,0), (2,2), (4,2),..., are the successive convergents of the continued fraction expansion[3,7]

$$\sqrt{1-s^2} = 1 - \frac{s^2}{2-}\,\frac{s^2}{2-}\,\frac{s^2}{2-}\cdots. \tag{10}$$

One readily sees that these approximations satisfy the recurrence relation

$$r^{(1)}(s) \equiv 1, \quad r^{(K+1)}(s) = 1 - s^2/[1 + r^{(K)}(s)], \quad K \geqslant 1. \tag{11}$$

They can also be represented explicitly by the formula

$$r^{(K)}(s) = 1 - s^2 U^*_{K-1}(s)/U^*_K(s), \tag{12}$$

where U_K is the Kth Chebyshev polynomial of the second kind, and U_K^* is its "reciprocal polynomial" $U_K^*(s) = s^K U_K(s^{-1})$ (see Ref. 20).

B. Interpolation in Chebyshev points

The simplest way to distribute interpolation points around $[-1,1]$ would be to place them at equal intervals, but this produces very poor approximations—the "Runge phenomenon."[21] A standard alternative choice is the set of *Chebyshev points* in $[-1,1]$, which amounts to equal spacing with respect to θ:

$$s_k = \sin\theta_k, \quad \theta_k = -\pi/2 + [\pi(k-\tfrac{1}{2})]/2K, \quad 1 \leqslant k \leqslant 2K. \tag{13}$$

The Chebyshev points are distributed more densely near $s = \pm 1$ than near $s = 0$, and indeed this is true of all of our approximations except Padé (see Fig. 5 below). The higher density near the end points is necessary for good wide-angle behavior because of the square-root singularities there.

We computed approximants of this kind by the general procedure (9).

C. Least-squares or L^2

A type (m,n) least-squares approximation to $\sqrt{1-s^2}$ is defined by the condition

$$\|\sqrt{1-s^2} - r(s)\|_2 = \left(\int_{-1}^{1} [\sqrt{1-s^2} - r(s)]^2\, ds\right)^{1/2} = \text{minimum}, \tag{14}$$

where the minimum is taken over all rational functions of type (m,n). Computing linear $(n=0)$ least-squares fits is easy, but in nonlinear cases one must resort to an iterative process. To do this, we again took advantage of the interpolation procedure of the last section, and chose $s_1,...,s_K$ as independent variables (rather than the coefficients of r) to be adjusted to minimize (14). At this point, one could make use of a general software product for multivariate nonlinear least-squares calculations, perhaps after first introducing a transformation from the constrained variables $0 < s_1 < \cdots < s_K < 1$ to unconstrained variables $\sigma_k \in (-\infty, \infty)$. Instead, we simply performed a sequence of univariate least-squares minimizations, optimizing $s_1 \in (0, s_2)$, $s_2 \in (s_1, s_3)$, and so on cyclically until convergence was achieved. This works very well in practice, and we have little doubt that our results represent unique best L^2 fits to $\sqrt{1-s^2}$, although we do not have a proof of this. (In general, nonlinear L^2 approximations are not unique, and have no simple characterization.[22])

One-way wave equations based on least-squares approximation have been computed previously by Lindman,[8] Bamberger *et al.*,[23] and Wagatha.[24] The first two of these use an L^2 problem defined slightly differently from ours, so their results are not directly comparable. Wagatha's least-squares approximations are the same as ours, but he considers only the cases (0,0) and (2,0) that can be determined analytically (see Table I).

The dashed line of Fig. 2 represents the least-squares one-way wave equation of type (2,0).

D. Chebyshev or L^∞

A type (m,n) Chebyshev or L^∞ approximation to $\sqrt{1-s^2}$ is defined by the L^∞ analog of (14),

$$\|\sqrt{1-s^2} - r(s)\|_\infty = \max_{s\in[-1,1]} |\sqrt{1-s^2} - r(s)| = \text{minimum}. \quad (15)$$

This condition is guaranteed to determine a unique function $r(s)$, which is characterized by an error curve that "equioscillates" between $m+n+2$ extreme values.[25,12] The two standard methods for computing rational L^∞ approximations are the Remes algorithm and the differential correction algorithm,[21] but once again, we have chosen instead a simpler approach based on the interpolation formula (9). As in the L^2 case, we performed a cyclical sequence of univariate computations, choosing $s_1 \in (0,s_2)$ to minimize the L^∞ norm in that interval, then $s_2 \in (s_1,s_3)$ to minimize the L^∞ norm there, and so on. The process converges quickly, and the construction guarantees that the resulting function has equiripple behavior and is therefore optimal.

E. Chebyshev on a subinterval, or L_α^∞

If the type (2,2) Padé approximant is regarded as a good fit for $\theta \in [-\alpha,\alpha]$ with $\alpha = 45°$, say, why not compute the actual best L^∞ approximant on that subinterval instead? The result will be a closer approximation in the L^∞ sense on both $[-\alpha,\alpha]$ and $[-90°,90°]$, and it can be calculated by the same cyclical iteration as above. This idea was first proposed by Greene.[9] In the experiments of the next section, we have chosen the following values:

K	(m,n)	α
1	(0,0)	10°
2	(2,0)	20°
3	(2,2)	45°
4	(4,2)	60°
5	(4,4)	75°

F. Interpolation in Newman points

Newman proved in 1964 that, whereas type $(n,0)$ Chebyshev approximations to $\sqrt{1-s^2}$ converge no faster than $O(n^{-1})$ in the L^∞ norm as $n\to\infty$, type (n,n) approximations have L^∞ error $O(e^{-C\sqrt{n}})$ for some $C>0$ (Ref. 15). The optimal constant was later shown to be $C=\pi$.[26] Newman's proof is based on a clever choice of interpolation points for which the error can be shown to be $O(e^{-\sqrt{n}})$. One reason we consider interpolation in these "Newman points" here is that it provides an easy and explicit method for obtaining near-best one-way wave equations in the L^∞ sense. Another is that Newman's idea gives insight about the clustering of interpolation points near $s=\pm 1$ for other high-accuracy approximations, even though his distribution of points is not precisely optimal.

The Newman points are defined by

$$s_1 = 1, \quad s_k = \sqrt{1-\xi^{k-2}}, \quad 2\leqslant k\leqslant K, \quad (16)$$

where

$$\xi = e^{-1/\sqrt{K-1}}.$$

Thus they are approximately geometrically distributed with respect to θ near $\pm 90°$. One should bear in mind that Newman introduced these points only for their asymptotic behavior as $K\to\infty$, so it might very well make sense to modify the definition in some way to get better behavior for small K.

G. Chebyshev–Padé

Finally, Chebyshev–Padé approximation, introduced in the 1970s by Clenshaw and Lord and by Gragg,[27,28] is an analog of Padé approximation designed for the interval $[-1,1]$ rather than the point $s=0$. Let $\sqrt{1-s^2}$ be expanded in the Chebyshev series

$$\sqrt{1-s^2} = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{1}{3}T_2(x) + \frac{1}{15}T_4(x) + \frac{1}{35}T_6(x) + \cdots\right), \quad (17)$$

where T_k is the kth Chebyshev polynomial of the first kind (see Ref. 25). The type (m,n) Chebyshev–Padé approximation to $\sqrt{1-s^2}$ is the unique rational function of that type whose Chebyshev expansion matches (17) up to order $m+n+2$:

$$r(s) - \sqrt{1-s^2} = O[T_{m+n+2}(s)].$$

This is the same as the definition of a Padé approximation, but with the Taylor series replaced by the Chebyshev series. The motivation behind it is that one can expect good approximation throughout $[-1,1]$, but without the need for the iterative calculations involved in L^2 and L^∞ approximation.

We computed coefficients of Chebyshev–Padé approximations by a standard procedure due to Gragg.[28] If z is a complex number on the circle $|z|=1$, with $s = \mathrm{Re}\, z = \frac{1}{2}(z+z^{-1})$, then (17) can be rewritten

$$\sqrt{1-s^2} = (2/\pi)\mathrm{Re}\, h(z) = (1/\pi)[h(z) + h(z^{-1})] \quad (18)$$

with

$$h(z) = \frac{1}{2}\left(z - \frac{1}{z}\right)\log\left(\frac{1+z}{1-z}\right) = 1 - \tfrac{2}{3}z^2 - \tfrac{2}{15}z^4 - \tfrac{2}{35}z^6 - \cdots. \quad (19)$$

Let $R(z)$ be the type (m,n) Padé approximation to (19), which is equal to the Kth convergent of the continued fraction

$$h(z) = 1 - \frac{2z^2}{3-}\,\frac{3z^2}{5-}\,\frac{8z^2}{7-}\,\frac{15z^2}{9-}\,\frac{24z^2}{11-}\cdots. \quad (20)$$

Then

$$r(s) = (2/\pi)\mathrm{Re}\, R(z) = (1/\pi)[R(z) + R(z^{-1})] \quad (21)$$

is the Chebyshev–Padé approximation of the same type to $\sqrt{1-s^2}$.

This is the only one of our seven approximations for

TABLE I. One-way wave equations of orders 1 and 2.

	$K=1$ (0,0)	$K=2$ (2,0)
Padé	$u_x = u_t$	$u_{xt} = u_{tt} - \frac{1}{2} u_{yy}$
Chebyshev points	$u_x = (1/\sqrt{2})u_t$	$u_{xt} = (3\gamma - 2\gamma^3)u_{tt} - 2\gamma u_{yy}, \quad \gamma = \sin(\pi/8)$
L^2	$u_x = (\pi/4)u_t$	$u_{xt} = (21\pi/64)u_{tt} - (15\pi/64)u_{yy}$
Chebyshev–Padé	$u_x = (2/\pi)u_t$	$u_{xt} = (10/3\pi)u_{tt} - (8/3\pi)u_{yy}$
Newman points	$u_x = 0$	$u_{xt} = u_{tt} - u_{yy}$
L^∞	$u_x = \frac{1}{2} u_t$	$u_{xt} = \frac{9}{8} u_{tt} - u_{yy}$

which the points of interpolation are not given directly by the calculation procedure. We determined them by applying a polynomial rootfinder to the equation $q^2(s)(1 - s^2) = p^2(s)$.

H. Other approximations

Seven families of approximations may already seem like four or five too many, and yet there are many further possibilities that also make sense. Two general procedures for rational approximation that we have not discussed are Carathéodory–Fejér approximation[29] and rational approximation with predetermined pole locations via sinc functions.[30] There are also many interesting variations on the schemes we have mentioned, such as L^2 approximation on subintervals, weighted L^2 approximation, and (a special case) L^2 approximation with respect to θ. Also, one might choose to compute an L^2 approximation, say, constrained to have two or more points of interpolation at $s = 0$.[23]

III. NUMERICAL RESULTS

There are many different ways to compare approximate one-way wave equations. For example, in underwater acoustics one may be interested in the error $r(s) - \sqrt{1-s^2}$, while in absorbing boundary conditions the reflection coefficient $[r(s) - \sqrt{1-s^2}]/[r(s) + \sqrt{1-s^2}]$ is more important. In what follows, we have been forced to be selective.

To begin the comparison, Table I lists the exact coefficients for six of our one-way wave equations of degrees 1 and 2 (all but L^∞_α, whose coefficients are messy). These are the linear cases of types $(m,0)$, where explicit coefficients are easily derived. It is interesting to note that even in approximation of degree 1, all six equations are distinct.

One way to examine how accurate these approximations are is to look at L^2 and L^∞ norms of the errors. Figure 3 plots these norms on a logarithmic scale for all seven of our approximations, and Tables II and III list the same results numerically. Notice that, as one would expect, the Padé approximants fare worst in both of these norms, because of their poor wide-angle behavior.

However, L^2 and L^∞ norms reveal fairly little. To improve on them, Fig. 4 plots the error $r(s) - \sqrt{1-s^2}$ as a function of $\theta = \sin^{-1} s$ for $1 \leqslant K \leqslant 5$. (Each plot is scaled independently, and the large values near $\theta = 90°$ have been clipped.) These plots show a great deal more than Fig. 3 and Tables II and III, and, in particular, we make the following observations:

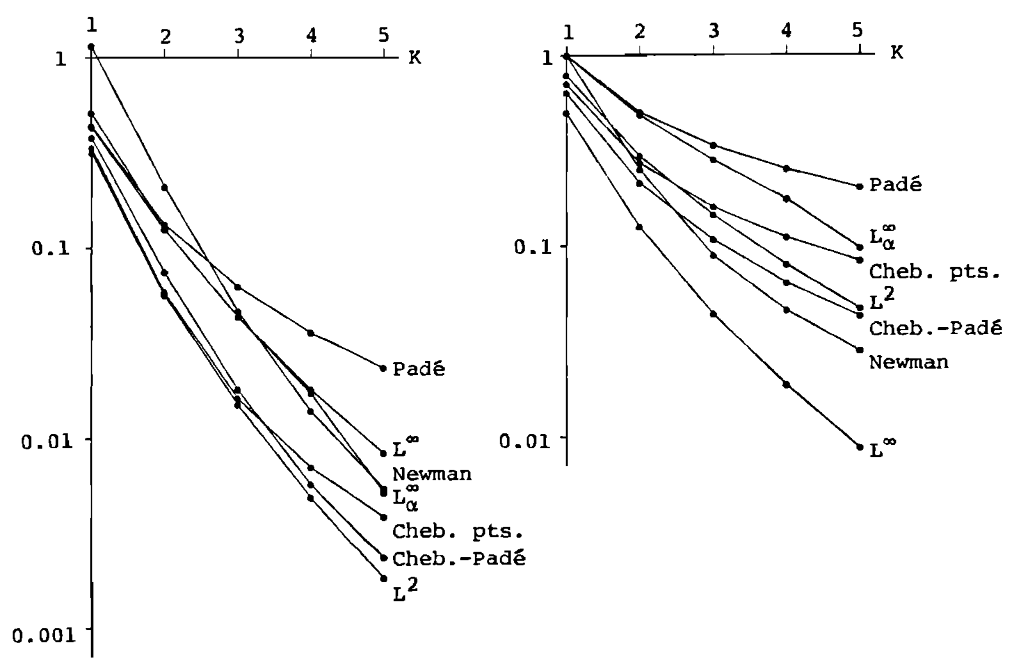
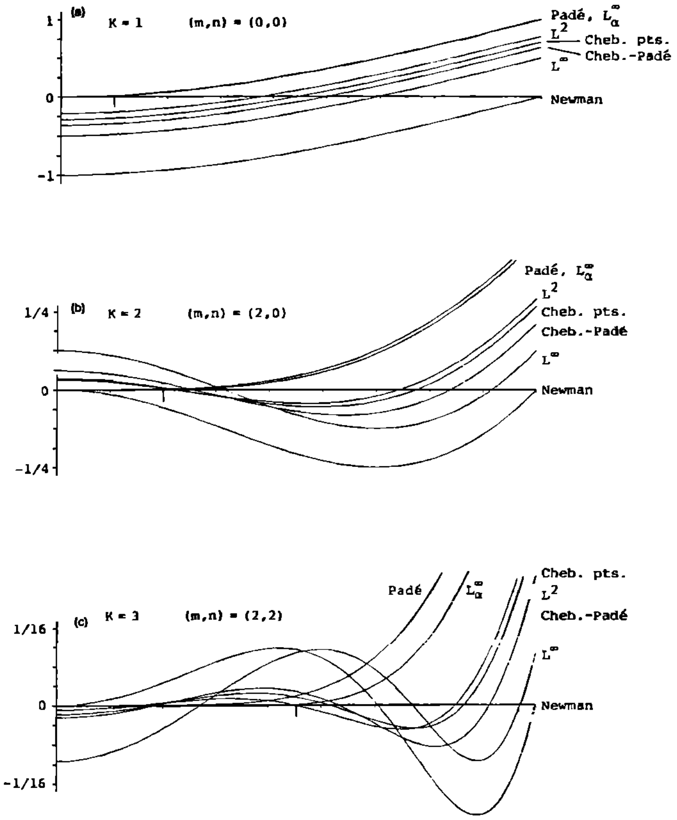


FIG. 3. L^2 and L^∞ errors as functions of K.



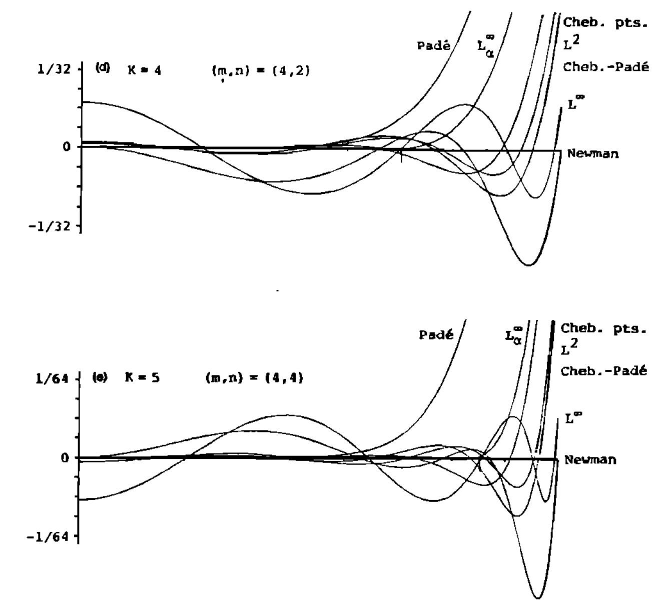


FIG. 4. Errors $r(s) - \sqrt{1-s^2}$ as functions of $\theta = \sin^{-1} s$ for one-way wave approximations of orders 1–5. The horizontal axis represents θ marked in units of 10° from 0° to 90°. The long tick mark in each plot is the angle α for the L_α^∞ fit.

(1) Clearly, it is much more difficult for an approximation to be accurate near $\theta = 90°$ than near $\theta = 0°$.

(2) The Padé approximants are by far the best near $\theta = 0°$, and by far the worst near $\theta = 90°$.

(3) At the other extreme, the L^∞ approximant has equiripple behavior throughout $[-90°, 90°]$, but pays a great price for this near $\theta = 0°$.

(4) Chebyshev interpolation, L^2 approximation, and Chebyshev–Padé approximation seem to offer attractive compromises between narrow-angle and wide-angle accuracy.

(5) For practical purposes, the L_α^∞ approximations are as good as or better than Padé for all $\theta \in [-90°, 90°]$. Of course, the Padé approximants are slightly more exact near $\theta = 0°$, but their accuracy there is probably unobservable in applications.

To substantiate this last remark, Table IV lists L^∞ errors on both $[-\alpha, \alpha]$ and $[-90°, 90°]$ for the Padé and L_α^∞ approximations of degrees $K = 1,...,5$. The L_α^∞ approximations are considerably better on the subinterval—errors well under 0.1% except in the case $K = 1$—and yet they are better globally, too. In the light of these numbers, it is hard to see any advantage to Padé approximations, except simplicity.

In all of these approximations there is a strong correlation between low accuracy near $\theta = 0°$, high accuracy near $\theta = 90°$, and concentration of interpolation points near there. Figure 5 illustrates this phenomenon by displaying the interpolation points for the various approximants in the case $K = 5$.

It remains to list the coefficients of our various approximations. We will do this in two ways. First, Table V presents the interpolation points in $\theta \in [0°, 90°]$ for each of the seven approximations and for $1 \leqslant K \leqslant 5$. These numbers can be used

TABLE II. L^2 errors for various one-way wave equations.

	$K=1$ (0,0)	$K=2$ (2,0)	$K=3$ (2,2)	$K=4$ (4,2)	$K=5$ (4,4)
Padé	0.43788	0.13332	0.06277	0.03617	0.02342
L_α^∞	0.43051	0.12494	0.04375	0.01731	0.00520
Chebyshev points	0.33450	0.05866	0.01629	0.00708	0.00389
L^2	0.31565	0.05701	0.01514	0.00491	0.00181
Chebyshev–Padé	0.37935	0.07488	0.01812	0.00574	0.00234
Newman points	1.15470	0.20930	0.04670	0.01394	0.00543
L^∞	0.51238	0.12524	0.04353	0.01776	0.00850

TABLE III. L^∞ errors for various one-way wave equations.

	$K=1$ (0,0)	$K=2$ (2,0)	$K=3$ (2,2)	$K=4$ (4,2)	$K=5$ (4,4)
Padé	1.00000	0.50000	0.33333	0.25000	0.20000
L_α^∞	0.99240	0.48469	0.27961	0.17423	0.09553
Chebyshev points	0.70711	0.27060	0.15840	0.10931	0.08234
L^2	0.78540	0.29452	0.14344	0.07875	0.04651
Chebyshev–Padé	0.63662	0.21221	0.10610	0.06366	0.04244
Newman points	1.00000	0.25000	0.08814	0.04564	0.02782
L^∞	0.50000	0.12500	0.04377	0.01852	0.00850

TABLE IV. Comparison of L^∞ errors of Padé and L^∞_α approximations.

		$\|r(s)-\sqrt{1-s^2}\|_{[-\alpha,\alpha]}$		$\|r(s)-\sqrt{1-s^2}\|_{[-90^\circ,90^\circ]}$	
K	α	Padé	L^∞_α	Padé	L^∞_α
1	10°	0.01519	0.00760	1.00000	0.99240
2	20°	0.00182	0.00023	0.50000	0.48469
3	45°	0.00718	0.00027	0.33333	0.27961
4	60°	0.01250	0.00015	0.25000	0.17423
5	75°	0.03942	0.00024	0.20000	0.09553

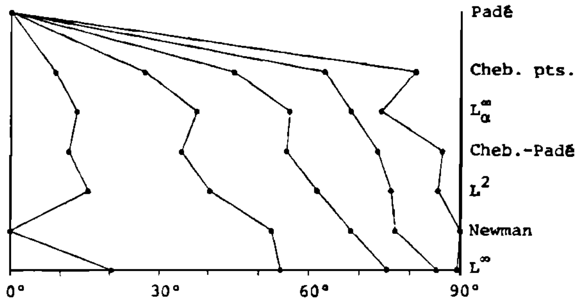


FIG. 5. Interpolation points in [0°,90°] for various approximations of order $K = 5$, type (4,4).

TABLE V. Interpolation points in [0°,90°] (degrees).

K	Padé	L^∞_α	Chebyshev points	L^2	C–P	Newman points	L^∞
1	0.000	7.067	45.000	38.242	50.460	90.000	60.000
2	0.000	7.620	22.500	22.137	25.781	0.000	31.396
	0.000	18.464	67 500	64.416	73.881	90.000	81.579
3	0.000	11.692	15.000	18.372	18.429	0.000	26.877
	0.000	31.859	45.000	51.327	53.147	60.458	66.641
	0.000	43.469	75.000	76.607	81.421	90.000	87.015
4	0.000	12.180	11.250	15.845	14.237	0.000	22.574
	0.000	34.245	33.750	44.977	41.832	55.848	59.742
	0.000	50.426	56.250	67.389	66.274	71.630	79.795
	0.000	58.918	78.750	82.484	84.711	90.000	88.734
5	0.000	13.274	9.000	15.601	11.625	0.000	20.385
	0.000	37.508	27.000	40.418	34.386	52.661	54.441
	0.000	56.150	45.000	61.656	55.553	68.415	75.497
	0.000	68.270	63.000	76.303	73.531	77.107	85.310
	0.000	74.250	81.000	85.527	86.424	90.000	89.419

TABLE VI. Coefficients of one-way wave equations in the representation (22).

K	Padé	L^∞_α	Chebyshev points	L^2	C–P	Newman points	L^∞
1	1.00000	0.99240	0.70711	0.78540	0.63662	0.00000	0.50000
2	1.00000	1.00023	1.03597	1.03084	1.06103	1.00000	1.12500
	− 0.50000	− 0.51555	− 0.76537	− 0.73631	− 0.84883	− 1.00000	− 1.00000
3	1.00000	0.99973	0.99650	0.99250	0.99030	1.00000	0.95651
	− 0.75000	− 0.80864	− 0.91296	− 0.92233	− 0.94314	− 1.00000	− 0.94354
	− 0.25000	− 0.31657	− 0.47258	− 0.51084	− 0.55556	− 0.66976	− 0.70385
4	1.00000	1.00015	1.00034	1.00227	1.00161	1.00000	1.01773
	− 1.00000	− 1.16394	− 1.27073	− 1.37099	− 1.37170	− 1.48698	− 1.59644
	0.12500	0.22308	0.29660	0.38178	0.38027	0.48698	0.57976
	− 0.50000	− 0.65974	− 0.76017	− 0.83407	− 0.84000	− 0.91384	− 0.94301
5	1.00000	0.99977	0.99997	0.99904	0.99973	1.00000	0.99151
	− 1.25000	− 1.60852	− 1.58426	− 1.73635	− 1.70184	− 1.81757	− 1.87681
	0.31250	0.61776	0.59248	0.73920	0.70421	0.81757	0.88536
	− 0.75000	− 1.11408	− 1.08588	− 1.25338	− 1.20988	− 1.35352	− 1.46935
	0.06250	0.20832	0.18537	0.29403	0.25926	0.37204	0.47568

together with (9) to generate the rational functions and hence one-way wave equations in question, and they are also interesting in their own right.

Next, we list the coefficients themselves in Table VI. Let $r(s)$ be given in the form

$$r(s) = \frac{p(s)}{q(s)} = \sum_{j=0}^{m/2} a_j s^{2j} \Big/ \sum_{j=0}^{n/2} b_j s^{2j}, \quad b_0 = 1. \tag{22}$$

The coefficients are listed in the pattern

$$\begin{array}{c} a_0 \\ a_1 \\ \vdots \\ a_{m/2} \\ \hline b_1 \\ \vdots \\ b_{n/2}. \end{array}$$

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