Instability of Difference Models for Hyperbolic Initial Boundary Value Problems

LLOYD N. TREFETHEN
Courant Institute

Abstract

A theory of instability is presented for finite difference models of linear hyperbolic partial differential equations in one space dimension with a boundary. According to this theory, instability is caused by spurious radiation of wave energy from the boundary at a numerical group velocity $C \geqq 0$. To make this point of view precise, we first develop a rigorous description of group velocity for difference schemes and of reflection of waves at boundaries. From these results we then obtain lower bounds for growth rates of unstable finite difference solution operators in l^2 norms, which extend earlier results due to Osher and to Gustafsson, Kreiss, and Sundström. In particular we investigate l^2-instability with respect to both initial and boundary data, and show how they are affected by (a) finite versus infinite reflection coefficients and (b) wave radiation with $C = 0$ versus $C > 0$.

1. Introduction

In solving linear hyperbolic partial differential equations numerically by means of finite difference approximations, a principal difficulty both theoretically and in practice is the question of stability. For the "Cauchy problem" on the unbounded domain $(-\infty, \infty)$, a fairly complete stability theory based on Fourier analysis has been worked out during the last few decades by von Neumann, Lax, Kreiss, and others [1], [14], [20], [22]. For the "initial boundary value problem" on a domain such as $[0, \infty)$ or $[0, 1]$, however, Fourier analysis cannot be applied in a straightforward way, and progress has been slower and technically more complex. The most important contributions in this area were made around 1970 by S. Osher [18], [19] and by H.-O. Kreiss and his colleagues [5], [9], [10], and are based on various kinds of *normal mode analysis* that extend the Fourier methods. A comprehensive theory of this type was presented in 1972 in an influential paper by Gustafsson, Kreiss, and Sundström [5].

The stability criterion derived in the Kreiss–Osher theory involves a "perturbation test" for "generalized eigensolutions", whose meaning has in the past been obscure. But recently we have pointed out (see [25]) that this test has a physical interpretation in terms of *group velocity*, a fundamental concept in the theory of dispersive waves. In a dispersive system, the group velocity for waves of frequency ω and wave number ξ is defined as $C = -d\omega/d\xi$, where ω and ξ are related by a *dispersion relation* $p(\omega, \xi) = 0$, and it can be shown that energy associated with the wave component $e^{i(\omega t + \xi x)}$ propagates asymptotically (in various senses) at

Communications on Pure and Applied Mathematics, Vol. XXXVII, 329–367 (1984)
 CCC 0010-3640/84/030329-39$04.00

this speed (see [15], [29]). In particular this applies to finite difference models of hyperbolic equations, because discretization inevitably introduces dispersion. The group velocity interpretation of the Kreiss–Osher results is roughly this: a difference model is unstable if and only if it permits spontaneous radiation of energy from the boundary into the interior of the mesh – that is, for the case of a boundary at the left, if it admits a solution consisting of a set of waves all of which have $C \geqq 0$. Such waves are in general "numerical parasites" rather than physically meaningful solutions to the original differential equation.

The first purpose of this paper is to carry out a systematic study of the dispersive properties of difference models, so that the group velocity interpretation of the instability problem can be made precise. We hope that the same results will also prove useful for other problems in the analysis of difference models. First we show that every wave $e^{i(\omega t+\xi x)}$, $\xi, \omega \in \mathbf{R}$, admitted by a Cauchy stable difference model has a (real) group velocity, even if the model is dissipative (Lemma 3.2). On the basis of this result we then develop a classification of so-called *steady-state solutions* into *leftgoing, strictly leftgoing, rightgoing*, and *strictly rightgoing* components, corresponding roughly to group velocities $C \leqq 0$, $C < 0$, $C \geqq 0$, and $C > 0$ (Definitions 3.2, 3.3, 4.1, 4.2). These definitions reduce the main theorem of [5] to the following simple form:

THEOREM 1a. (GKS stability theorem). *A difference model of an initial boundary value problem (on the domain* $[0, \infty)$*) is GKS-unstable if and only if it admits a rightgoing steady-state solution.*

Theorem 1a is not as simple as it looks, however, because it depends on the rather complicated notion of *GKS-stability* (Definition 4.6), which is discussed at the end of this section and in Section 4.

Our second purpose is to derive growth estimates for solutions to unstable difference models which reflect in a natural way the mechanisms that underlie numerical instability. There are several of these mechanisms, and they differ widely in strength. The GKS stability definition is strict enough to encompass all of them, and as a result, Theorem 1a asserts fairly little about what growth must be exhibited by solutions to unstable difference models. But we show that if a small class of marginally GKS-unstable borderline cases is excluded, then instability must also occur in the l^2 norm. One price paid for these results is the sacrifice of the "sufficient" half of the necessary and sufficient condition for stability of Theorem 1a. Certain sufficient conditions for l^2-stability have however been obtained in the past (see [9], [10], [18], [19]).

In particular we investigate two distinctions that are not present in the theories of Kreiss, et al. or Osher. The first is the distinction between *rightgoing* ($C \geqq 0$) and *strictly rightgoing* ($C > 0$) *steady-state solutions* for the initial boundary value problem model (Definitions 4.1, 4.2). By exhibiting one of these borderline models that admits a GKS-unstable solution with $C = 0$ but is stable in l^2, we show:

THEOREM 1b. *GKS-instability does not imply l^2-instability.*

On the other hand, for $C>0$ one gets a definite rate of growth in l^2:

THEOREM 2a. *The existence of a strictly rightgoing steady-state solution implies l^2-instability, with an unstable growth rate at least proportional to $\sqrt{n}$.*

(Here n denotes the number of time steps. Moreover, h will denote the spatial grid size.) The idea behind Theorem 2a is illustrated in Figure 1. If a difference model admits a solution consisting of a wave with $C>0$, consider initial data consisting of a few wave lengths of this oscillation near the boundary and zero elsewhere. As t increases the wave front will propagate into the field at speed $C>0$, causing growth in the l^2 norm by $t=1$ on the order of $O(1/\sqrt{h})=O(\sqrt{n})$. Unstable growth at the rate $\sqrt{n}$ has been observed previously by various authors. See Section 3 of [6], where the Kreiss–Osher ideas are applied to examine behavior on either side of a shock, and also Section 17 of [13].

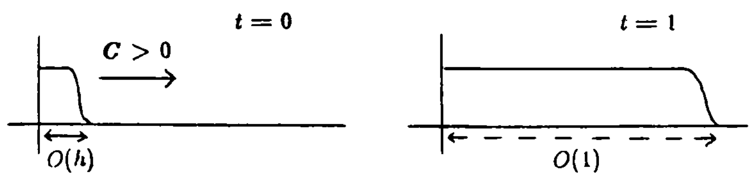

Figure 1. Instability caused by radiation of a rightgoing wave from the boundary.

The second distinction we introduce is between *finite* and *infinite reflection coefficients* associated with unstable steady-state solutions (Definition 4.3). If a difference model admits a rightgoing steady-state solution, then usually there is a set of corresponding leftgoing solution modes whose reflection coefficient matrix at the boundary is formally infinite. But there are also realistic problems where this is not so, and we show that the difference is significant to the unstable growth rate:

THEOREM 3a. *The existence of a strictly rightgoing steady-state solution with an infinite reflection coefficient implies that the l^2-unstable growth rate is at least proportional to n.*

The idea behind Theorem 3a is illustrated in Figure 2. If initial data are taken consisting of a leftgoing pulse at the critical frequency, the result will be a reflected signal whose amplitude is larger by a factor on the order of $O(1/h)=O(n)$ (the closest one can come to infinity on a discrete mesh), causing a growth in the l^2 norm of the same order.

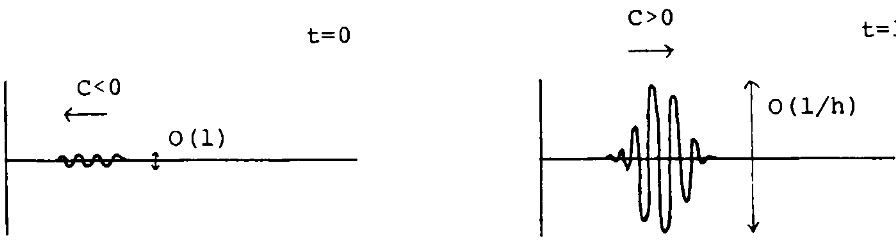

Figure 2. Instability caused by reflection in the boundary of a wave with infinite reflection coefficient.

Theorems 2a and 3a can be interpreted as follows: a strictly rightgoing steady-state solution is a kind of unstable resonance, but only if an infinite reflection coefficient is present will the resonance be strongly excited by initial data. On the other hand, we also consider instability with respect to boundary data, and show that here an infinite reflection coefficient is no longer needed to excite the instability:

THEOREM 5a. *The existence of a strictly rightgoing steady-state solution implies l^2-instability with respect to boundary data, with an unstable growth rate at least proportional to n.*

Our results are summarized in Table I. In each case Theorem Na, $N = 1, \cdots, 6$, establishes that the growth is at least proportional to a certain function of n, and Theorem Nb is the converse "sharpness" result that the growth is in general no faster. (We do not prove a result for position 4a. It is likely that

Table I
Summary of Unstable Growth Rates for Various Classes of Difference Models

	$\nexists$ rightg st-st soln	$\exists$ rightg st-st soln	$\exists$ str rightg st-st soln	$\exists$ str rightg st-st soln with inf. r.c.	$\exists$ str rightg st-st soln with $\lvert z\rvert > 1$ (Godunov-Ryabenkii)
growth wrt initial data	1	1 Thms 1a, 1b	$\sqrt{n}$ Thms 2a, 2b	n Thms 3a, 3b	const^n
growth wrt bndry data	1	$\sqrt{n}$ Thm 4b	n Thms 5a, 5b	n Thms 6a, 6b	const^n

← GKS-stable | GKS-unstable →

growth at the rate $\sqrt{n}$ does not occur for all models in this category, but only for those for which $dC(\xi)/d\xi \neq 0$ at the critical wave number.)

The theorems of Table I hold for general multilevel approximations to the scalar model equation $u_t = au_x$ (a constant), which may be explicit or implicit, dissipative or nondissipative. (Our proofs for Theorems 2a-6a treat two-level approximations only, but an outline of the extension of most of them to the multilevel case can be found in Appendix B of [24].) The results generalize readily to diagonalizable models of hyperbolic systems with constant coefficients (see [24]), and probably also to systems with Lipschitz continuous variable coefficients.

OUTLINE OF THE PAPER. For an introduction to group velocity effects in finite difference schemes, see [23] or [28], and for an informal presentation of the group velocity interpretation of the Kreiss–Osher theory, see [25]. Much of this material is also discussed at length in the author's PhD dissertation [24]. The present paper is organized as follows. Section 2 presents numerical experiments to confirm that the distinctions of Table I are significant in practice. This also serves as a concrete introduction to the results that follow. Section 3 proves the existence of the group velocity, and defines the terms leftgoing, rightgoing, and so on. Section 4 introduces a left-hand boundary at $x = 0$ and the required additional boundary conditions, describes reflections at this boundary, and sets forth various definitions of stability for initial boundary value problems. Section 5 proves the main theorems (Theorems 1a–6a) for the case of two-level models, and Section 6 establishes the corresponding sharpness results (Theorems 1b-6b).

A number of additional remarks must be made to clarify the relationship of the present theory to previous work, and other issues.

WELL-POSEDNESS FOR DIFFERENTIAL EQUATIONS. The stability problem for difference models with boundaries is closely analogous to the corresponding well-posedness problem for hyperbolic partial differential equations, for which a theory has been worked out by Kreiss and Sakamoto that leads to an algebraic condition much like the Kreiss–Osher condition described above (see [11], [12], [21]). It turns out that just as in the stability problem, this condition amounts to a requirement that no wave solutions exist which radiate spurious energy from the boundary. This interpretation of the theory is apparently due to A. Majda and his colleagues, and will be developed in a forthcoming paper by R. Higdon [7].

Well-posedness is inherently simpler than stability, however, because in the absence of a discrete grid, there is no wavelength-dependent dispersion. As a result ill-posedness only becomes an issue for problems in at least two space dimensions. The possibility of a wave traveling tangentially along a boundary then emerges as a troublesome borderline case, analogous to our waves with

$C=0$ (see [7]). For difference models in several dimensions all complications must be considered at once, but Michelson [16] has provided a theory for these problems under a simplifying assumption of dissipativity.

ALTERNATIVE NOTIONS OF STABILITY. Our emphasis on l^2 norms is not meant to imply that this is the only useful way to measure stability (see Section 4 for a discussion of the l^2-versus GKS-stability definitions). The ultimate purpose of any stability investigation is to ensure convergence to the correct solution, but in complicated problems it is sometimes not obvious what kind of convergence is desired and what kind of stability will ensure it. In particular, some models that are GKS- and l^2-unstable may be stable in a weak sense, i.e., with respect to appropriate discrete Sobolev norms, and in some applications this will be enough for a successful computation. See the discussion by Kreiss in [12] of generalized eigensolutions of "first kind" and "second kind", and of what estimates can be obtained for each.

ANALOGY TO RESULTS ON INSTABILITY IN L^p. Our interest in algebraic growth rates for unstable numerical solution operators is motivated in part by analogous theorems that exist for the problem of instability of l^2-stable difference models on $(-\infty,\infty)$ in L^p norms, $p\neq 2$. These results are the work of many researchers, and are systematically presented in the monograph of Brenner, Thomée, and Wahlbin [1]. Dispersion phenomena underlie the L^p theory too, and a heuristic derivation of L^p-unstable growth rates by a group velocity argument is given in [26].

GODUNOV-RYABENKII INSTABILITY. We have not mentioned "Godunov-Ryabenkii eigensolutions" (Table 1) (see [20]), which generate catastrophic exponential growth. Such instabilities are well understood and easily recognized, but in practice it is the more subtle wavelike cases concentrated on here that cause the most trouble. Our theorems absorb the Godunov-Ryabenkii case into the class of strictly rightgoing steady-state solutions.

TESTING FOR INSTABILITY. The results of this paper provide new insight and new estimates, but they offer little help with the problem of actually testing a difference model for instability, which still reduces to the algebraic condition of Kreiss and Osher. In practice this test may be difficult (see [3], [30]).

DISSIPATIVE DIFFERENCE MODELS. As suggested above, our theorems hold for dissipative as well as nondissipative difference models. The effect of dissipativity is to rule out all wavelike steady-state solutions that oscillate with non-zero wave number in x. This drastically reduces the number of potential unstable solutions to be tested for.

TWO-BOUNDARY PROBLEMS. It is well known that the fairly mild algebraic growth rates of Table 1 may increase to exponential when a second boundary is added (see [13]). The wave propagation point of view is helpful to understanding this process, and in particular, the distinction between finite and infinite reflection coefficients turns out to be critical (see [27]). In principle such exponential blow-up might also be brought about by the introduction of variable coefficients or lower-order terms, but it is not clear whether this is a problem in practice.

2. Numerical Illustrations

Let $u_t = u_x$ on $x, t \geqq 0$ be modeled for $x > 0$ by the leap frog formula

$$(2.1) \qquad Q: \quad v_j^{n+1} = v_j^{n-1} + \lambda(v_{j+1}^n - v_{j-1}^n),$$

where $\lambda < 1$ is the mesh ratio k/h (see the next section for notational details). For any z with $|z| \geqq 1$, the general solution to Q of the form $v_j^n = z^n \phi_j$ can be written

$$(2.2) \qquad v_j^n = a_l z^n \kappa_r^j + a_r z^n \kappa_i^j,$$

where κ_l and $\kappa_r = -1/\kappa_l$ are the two solutions κ of

$$(2.3) \qquad z - \frac{1}{z} = \lambda\left(\kappa - \frac{1}{\kappa}\right)$$

with $(\mathcal{Re}\,\kappa_r)(\mathcal{Re}\,z) \leqq 0 \leqq (\mathcal{Re}\,\kappa_l)(\mathcal{Re}\,z)$ and $|\kappa_r| \leqq 1 \leqq |\kappa_l|$. (Equation (2.2) must be modified in the confluent case $\kappa_l = \kappa_r = \pm i$.) The two terms in (2.2) represent leftgoing and rightgoing waves, respectively; in particular, for $|\kappa| = 1$, (2.3) leads to the expression

$$(2.4) \qquad C = -\frac{\kappa + 1/\kappa}{z + 1/z} = -\frac{\mathcal{Re}\,\kappa}{\mathcal{Re}\,z}, \qquad |\kappa| = 1,$$

for the *group velocity* of the wave $z^n \kappa^j$, so that $C \leqq 0$ for $\kappa = \kappa_l$ and $C \geqq 0$ for $\kappa = \kappa_r$ (see [23], [24]).

To obtain a numerical solution on the mesh $j, n \geqq 0$ starting from given initial data v_j^0, v_j^1, Q can be applied for $j, n \geqq 1$, but an additional one-sided numerical boundary condition is needed to provide the values v_0^{n+1}. The combination of Q with a boundary formula of this kind will be denoted $\bar{Q}$. We take $\lambda = \frac{1}{2}$ and consider four possibilities:

$$(2.5\alpha) \qquad \alpha: \; v_0^{n+1} = v_1^n,$$

$$(2.5\beta) \qquad \beta: \; v_0^{n+1} = v_1^{n-2},$$

$$(2.5\gamma) \qquad \gamma: \; v_0^{n+1} = \tfrac{1}{2}(v_0^n + v_2^n),$$

$$(2.5\delta) \qquad \delta: \; v_0^{n+1} = v_1^{n+1}.$$

Each of these can be viewed as the imposition of a reflection coefficient function relating a_r to a_l in (2.2), which in general will restrict the set of solutions $v_j^n = z^n \phi_j$

from a two- to a one-parameter family. To compute the reflection relation for scheme $\bar{Q}_\alpha$, one inserts (2.2) into (2.5α) to obtain

$$z(a_l + a_r) = a_l\kappa_l + a_r\kappa_r,$$

i.e.,

$$(2.6\alpha) \qquad \bar{Q}_\alpha: \quad a_r(\kappa_r - z) = -a_l(\kappa_l - z).$$

Similarly for the other schemes the results are

$$(2.6\beta) \qquad \bar{Q}_\beta: \quad a_r(\kappa_r - z^3) = -a_l(\kappa_l - z^3),$$

$$(2.6\gamma) \qquad \bar{Q}_\gamma: \quad a_r(1+\kappa_r^2 - 2z) = -a_l(1+\kappa_l^2 - 2z),$$

$$(2.6\delta) \qquad \bar{Q}_\delta: \quad a_r(\kappa_r - 1) = -a_l(\kappa_l - 1).$$

A *rightgoing steady-state solution to* $\bar{Q}_x$ is a solution $v_j^n = z^n\phi_j$ containing rightgoing energy only; that is, a function (2.2) with $a_l = 0$, $a_r \neq 0$ that satisfies both (2.1) and (2.5x). Clearly, such a solution can exist if and only if the coefficient of a_r in (2.6x) is zero for some z with $|z| \geqq 1$. From (2.3) one can calculate that this can occur for models β, γ, δ but not α. The results are summarized in Table II.

Table II
Rightgoing Steady-State Solutions for Models
$\bar{Q}_\alpha, \bar{Q}_\beta, \bar{Q}_\gamma, \bar{Q}_\delta$

	z	κ_r	κ_l	$C(z, \kappa_r)$	a_r/a_l
α			(none)		
β	$e^{\pm\pi i/6}$	$\pm i$	$\pm i$	0	-1
γ	1	-1	1	1	$-\frac{1}{3}$
δ	-1	1	-1	1	∞

As indicated in the table, the rightgoing steady-state solutions admitted by $\bar{Q}_\gamma$ and $\bar{Q}_\delta$ have $C > 0$, hence are *strictly rightgoing*, while those admitted by $\bar{Q}_\beta$ have $C = 0$. Models $\bar{Q}_\gamma$ and $\bar{Q}_\delta$ differ in that the latter has an *infinite reflection coefficient* $a_r/a_l = \infty$ corresponding to the rightgoing steady-state solution, while the former has $a_r/a_l < \infty$, because the zero coefficient of a_r in (2.6γ) at $z = \kappa_l = -\kappa_r = 1$ is balanced by a zero coefficient of a_l for the same parameter values.

Thus $\bar{Q}_\alpha$, $\bar{Q}_\beta$, $\bar{Q}_\gamma$, $\bar{Q}_\delta$ exemplify the first four columns of Table I. The following two sets of experiments give evidence that the distinctions listed in the table are significant in practice.

First, Figure 3 shows a set of computations driven by *random initial data*

$$v_j^n = \zeta_j^n, \qquad n = 0, 1,$$

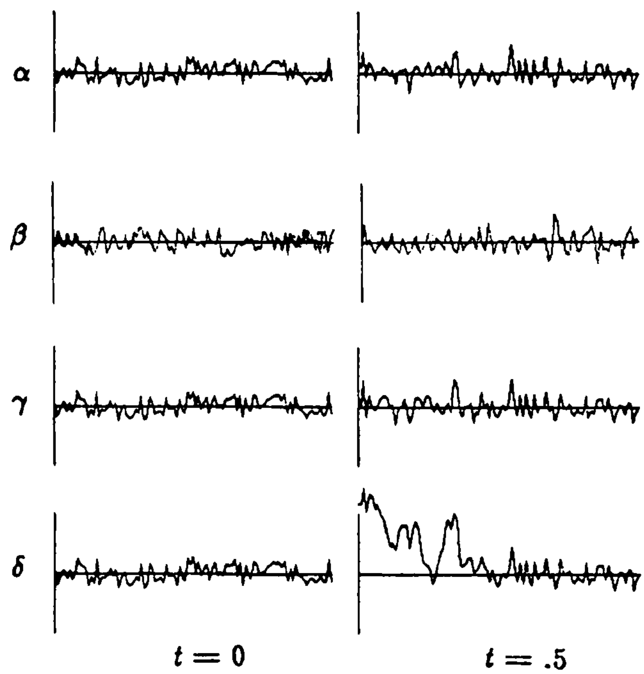

Figure 3. Schemes $\bar{Q}_\alpha - \bar{Q}_\delta$ with random initial data. Only $\bar{Q}_\delta$ is clearly unstable.

where $\{\zeta_j^n\}$ are uniformly distributed random numbers in $[-1, 1]$. The domain is $[0, 1]$, with mesh size $h = 1/100$ and right-hand boundary condition $v_{100}^n = 0$. For each scheme $\bar{Q}_\alpha$-$\bar{Q}_\delta$, the distributions v^0 and v^{100} $(t = \frac{1}{2})$ are plotted. Obviously an instability is present in case δ, where the first row of Table I predicts growth like n, while no instability is evident in cases α–γ, where the table predicts only $\sqrt{n}$. These results are consistent with the view that $\bar{Q}_\gamma$ and $\bar{Q}_\delta$ both admit unstable strictly rightgoing solutions, but only in the latter case is this mode strongly excited by initial data. (Of course, the lack of visible growth in Figures 3α–γ does not imply that $\bar{Q}_\gamma$ gives results that converge to the correct solution as the mesh is refined.)

Second, Figure 4 shows corresponding plots for zero initial data but *random boundary data*

$$g^n = \zeta^n$$

added as an inhomogeneous forcing term to the boundary condition (2.5); thus (2.5α) becomes

$$v_0^{n+1} = v_1^n + g^n,$$

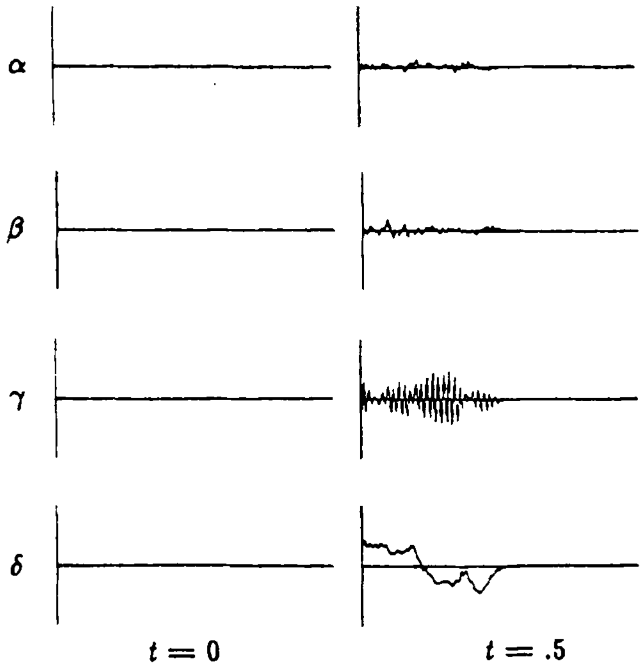

Figure 4. Schemes $\bar{Q}_\alpha - \bar{Q}_\delta$ with random boundary data. $\bar{Q}_\gamma$ and $\bar{Q}_\delta$ are clearly unstable.

and similarly for β, γ, δ. (Adding boundary data in this way is of course inappropriate for the differential equation $u_t = u_x$ under consideration, but this is beside the point; the same kind of behavior can occur in more realistic problems.) Now the situation has changed: while $\bar{Q}_\alpha$ and $\bar{Q}_\beta$ still appear stable, both $\bar{Q}_\gamma$ and $\bar{Q}_\delta$ are evidently unstable. In each case the boundary function has stimulated a rightgoing wave in the mode (z, κ_r) listed in Table II. The distinction between finite and infinite reflection coefficients has apparently ceased to matter, as predicted in the second row of Table I. (The growth visible in these plots in fact has order $O(\sqrt{n})$, but this would increase to $O(n)$ if the boundary data were not random but fixed at exactly the frequency of the unstable mode.)

One can devise many more experiments to explore the behavior of unstable difference models. In particular, the solution curves of Figures 3 and 4 were irregular because of the randomness in their forcing data, but smooth wavelike instabilities as in Figures 1 and 2 can also be observed numerically: see [27] and also Section 4.1 of [24]. Section 5 of [24] investigates borderline GKS-unstable models experimentally in some detail.

The proofs in Section 6 contain rigorous analyses of the instability of LF with several particular choices of boundary conditions, including δ.

3. Fundamentals for the Cauchy Problem

Consider the model hyperbolic equation

$$u_t = au_x, \qquad x, t \in \mathbf{R}, \tag{3.1}$$

where $a \in \mathbf{R}$ is constant and $u = u(x, t)$ is a complex-valued function to be determined. We postpone consideration of boundary and initial conditions to the next section. To approximate (3.1) by a finite difference model, we choose a *space step* $h > 0$ and a *time step* $k > 0$ and make use of a discrete formula to generate a grid function $\{v_j^n\}$, $j, n \in \mathbf{Z}$, which it is hoped will satisfy

$$v_j^n \approx u(jh, nk). \tag{3.2}$$

Let K and Z denote the shift operators

$$Kv_j^n = v_{j+1}^n, \qquad Zv_j^n = v_j^{n+1}. \tag{3.3}$$

We shall permit K or Z to act on any objects that have space or time indices, respectively. The approximation v will be computed as the solution of an $s+2$-*level finite difference formula*, denoted by Q, of the form

$$Q_{-1}v^{n+1} + \sum_{\sigma=0}^{s} Q_\sigma v^{n-\sigma} = 0, \qquad s \geqq 0, \tag{3.4}$$

where each Q_σ is a spatial difference operator

$$Q_\sigma = \sum_{j=-l}^{r} a_{j\sigma} K^j, \qquad a_{j\sigma} \in \mathbf{C}, \qquad -1 \leqq \sigma \leqq s. \tag{3.5}$$

Here l and r are fixed non-negative integers that indicate how far Q extends to the left and right of center.

In applying Q to generate v, one assumes at step $n+1$ that the values $v^{n-s}, \cdots, v^n$ are available. Let $\|\cdot\|$ denote the norm

$$\|\phi\|^2 = h \sum_{j=-\infty}^{\infty} |\phi_j|^2, \tag{3.6}$$

and let l^2 denote the set of spatial grid functions ϕ for which this number is finite. For (3.4) to be applicable for determining v^{n+1}, we assume that Q_{-1} has a bounded inverse as an operator on l^2.

In practice, Q may be applied with varying values of h and k, and it is hoped that as $h, k \to 0$, the solution v will approximate u, as suggested in (3.2). However, our stability results will make no reference to u, and (3.1) and (3.2) are provided for motivation only. We assume that although h and k may vary, the *mesh ratio* $\lambda = k/h$ is constant. This is typical for models of hyperbolic problems. We further assume that the coefficients $a_{j\sigma}$ of (3.5) may depend on λ, but not on h and k independently. Thus Q is a fixed difference formula, independent of x, t, h, and k. For such a formula the definition of stability takes the following simple form:

DEFINITION 3.1. Q is *Cauchy stable* if there exists a constant M such that, for any solution v of Q,

$$(3.7)\qquad \|v^n\| \leqq M \sum_{\sigma=0}^{s} \|v^\sigma\| \quad \text{for all} \quad n \geqq 0.$$

We assume always that this condition holds:

CAUCHY STABILITY ASSUMPTION. *Q is Cauchy stable.*

Q can be represented in various ways in terms of K and Z. Let P denote the bivariate polynomial

$$(3.8)\qquad P(\kappa, z) = \sum_{j=-l}^{r} \sum_{\sigma=-1}^{s} a_{j\sigma} \kappa^{l+j} z^{s-\sigma}.$$

Equation (3.4) can then be rewritten as

$$(3.9)\qquad P(K, Z) v_j^n = 0.$$

The *dispersion relation* for Q is the equation

$$(3.10)\qquad P(\kappa, z) = 0,$$

whose solutions are pairs $(\kappa, z) \in \mathbf{C}^2$. Whenever a relationship between κ and z is mentioned in what follows, it should be understood that we are speaking only of pairs (κ, z) that satisfy (3.10). For any non-zero values of κ and z the corresponding *wave number* ξ and *frequency* ω are defined modulo $2\pi/h$ and $2\pi/k$, respectively, by the formulas

$$(3.11)\qquad \kappa = e^{i\xi h}, \qquad z = e^{i\omega k}.$$

Whenever κ and ξ or z and ω are mentioned below, it is assumed that they are related by (3.11). Obviously $\xi \in \mathbf{R}$ if and only if $|\kappa| = 1$, and $\omega \in \mathbf{R}$ if and only if $|z| = 1$.

For any $z \in \mathbf{C}$, the associated *resolvent polynomial* is the univariate polynomial P_z of degree at most $l + r$ defined by

$$(3.12)\qquad P_z(\kappa) = P(\kappa, z).$$

In addition we let P_∞ denote the limiting polynomial associated with the coefficients at step $n + 1$ alone,

$$P_\infty(\kappa) = \sum_{j=-l}^{r} a_{j,-1} \kappa^{l+j} = \lim_{z \to \infty} P_z(\kappa)/z^{s+1}.$$

The *resolvent equation* for Q is the spatial difference operator equation

$$(3.13)\qquad P_z(K)\phi = 0.$$

We shall be concerned with the class of solutions v of Q of the form

$$v_j^n = z^n \phi_j \tag{3.14}$$

for some $z \in \mathbf{C}$ and some spatial grid function ϕ. (We do not require $\phi \in l^2$.) It is obvious that a distribution (3.14) is a solution to Q if and only if ϕ satisfies (3.13). For definiteness we make the following assumption of nondegeneracy. The first statement ensures that the stencil of Q extends a full l and r points to the left and right (cf. Assumption 5.5 of [5]), and the second that the center point $j=0$ is meaningfully defined (cf. Assumption 3.1 of [5] and the "pole condition" of [8]).

NONDEGENERACY ASSUMPTION. *For all $z \in \mathbf{C}$ with $|z| \geqq 1$, $P_z(\kappa)$ has non-zero 0-th and $(l+r)$-th coefficients. Moreover, $P_\infty(\kappa) \neq 0$ for $|\kappa| = 1$, and the curve P_∞ $(|\kappa| = 1)$ has positive winding number l about the origin.*

(The condition on $P_\infty(\kappa)$ actually follows by Fourier analysis from the invertibility of Q_{-1}.)

The following lemma describes all solutions (3.14) admitted by Q (cf. [5], pp. 659–660):

LEMMA 3.1. *For any $z \in \mathbf{C}$ with $|z| \geqq 1$, $P_z(\kappa)$ has exactly $l+r$ roots, counted with multiplicity, and they are non-zero. For $|z| > 1$, l of these satisfy $|\kappa| < 1$, r satisfy $|\kappa| > 1$, and none satisfy $|\kappa| = 1$. Given $|z| \geqq 1$, let $\{\kappa_i\}_{i \leqq 1 \leqq \mu}$, $\mu \geqq 1$, denote the distinct roots of $P_z(\kappa)$, and let v_i denote the multiplicity of κ_i. Then the $l+r$ sequences*

$$\phi_j = \kappa_i^j j^\delta, \qquad 1 \leqq i \leqq \mu, \quad 1 \leqq \delta \leqq \nu_i - 1, \tag{3.15}$$

are linearly independent solutions to (3.13), *and they span the linear space of all such solutions.*

Proof: The first part of the nondegeneracy assumption implies that P_z has $l+r$ roots, counted with multiplicity, and that they are all non-zero. For $|z| > 1$, a root with $|\kappa| = 1$ would contradict the von Neumann condition, hence Cauchy stability, and this implies that, for $|z| > 1$, the roots divide into well-defined sets inside and outside of the unit circle. The fact that there are l roots in the first set and r in the second can be established by examining the limit $z \to \infty$ and applying the winding number part of the nondegeneracy assumption (cf. Lemma 5.2 of [5]). Finally, the breakdown into fundamental solutions (3.15) is a standard result in the study of recurrence relations; see for example Section 4.2 of [17].

Suppose now that for some ξ_0, $\omega_0 \in \mathbf{R}$, i.e., $|\kappa_0| = |z_0| = 1$, Q admits a solution

$$v_j^n = z_0^n \kappa_0^j = \exp\{i(\omega_0 t + \xi_0 x)\}, \qquad x = jh, \quad t = nk. \tag{3.16}$$

If ω depends smoothly on ξ in a neighborhood of ξ_0, then according to the theory of dispersive waves (see [2], [15], [29]), the energy associated with the wave (3.16) travels at the *group velocity* C defined by the formula

$$C = -\frac{d\omega}{d\xi}\bigg|_{\xi_0,\omega_0}, \tag{3.17}$$

provided this number is real. By differentiating (3.11), one obtains the formulas

$$d\kappa = ih\kappa\, d\xi, \qquad dz = ikz\, d\omega,$$

and it follows that an equivalent expression for the group velocity is

$$C = -\frac{1}{\lambda}\frac{\kappa_0}{z_0}\frac{dz}{d\kappa}\bigg|_{\kappa_0,z_0}. \tag{3.18}$$

In Lemma 5.1 we shall justify in one of many possible ways the claim that C represents the velocity of energy propagation.

In the remainder of this section, we explore some of the purely algebraic properties of C. First, the following lemma establishes that C always exists when ξ_0 and ω_0 are real. This result holds not only for unitary approximations such as leap frog, Crank–Nicolson, and the Box scheme, but also for dissipative or partially dissipative formulas such as Lax–Wendroff, Lax–Friedrichs, and the Upwind formula (see Appendix A of [24]). Indeed we have made no assumptions regarding the dissipativity of Q.

LEMMA 3.2. *Suppose that Q admits the solution* (3.16) *with* $|\kappa_0| = |z_0| = 1$, *i.e.*, $\xi_0, \omega_0 \in \mathbf{R}$. *Then*,

(i) *in a neighborhood of* (κ_0, z_0), *z is a single-valued analytic function of κ*,

(ii) *the group velocity derivative* (3.17) *exists and is real*,

(iii) $C = 0$ *if and only if κ_0 is a multiple root of P_{z_0}. Therefore if $C \neq 0$, $z = z(\kappa)$ has an analytic inverse $\kappa = \kappa(z)$ near* (κ_0, z_0).

Proof: If Q admits the solution (3.16), then $P(\kappa_0, z_0) = 0$, where P is the polynomial defined by (3.8). For any $\kappa \in \mathbf{C}$, consider the univariate polynomial $P_\kappa(z)$ defined (cf. (3.12)) by

$$P_\kappa(z) = P(\kappa, z).$$

The coefficient of P_κ of degree $s+1$ is $P_\infty(\kappa)$, which by the Nondegeneracy Assumption is non-zero for $|\kappa| = 1$. In particular, P_κ must have exact degree $s+1$ for $\kappa = \kappa_0$, and since the coefficients of P_κ are analytic functions of κ, also in a neighborhood of κ_0. Now z_0 must be a simple root of P_{κ_0}, for if it were not, Q would admit a linearly growing solution

$$v_j^n = nz_0^n\kappa_0^j,$$

as in the proof of Lemma 3.1 but with the roles of j and n interchanged, and Fourier analysis shows that this would contradict Cauchy stability. These facts together with the implicit function theorem imply that, in a neighborhood of (κ_0, z_0), the dispersion relation $P(\kappa, z) = 0$ determines a single-valued analytic function $z(\kappa)$, satisfying

$$z - z_0 = A(\kappa - \kappa_0)^{\nu} + O((\kappa + \kappa_0)^{\nu+1}), \qquad A \neq 0, \tag{3.19}$$

for some $A \in \mathbf{C}$, where $\nu \geqq 1$ is the multiplicity of κ_0 as a root of P_{z_0}. This proves (i).

Claim (iii) follows from (3.18) and (3.19).

To establish (ii) we must show that C is real. For $\nu \geqq 2$ this is immediate; so assume $\nu = 1$. The situation is indicated in Figure 5: $z(\kappa)$ maps a neighborhood of κ_0 conformally onto a neighborhood of z_0. Now Cauchy stability implies that, for any κ with $|\kappa| = 1$, one must have $|z(\kappa)| \leqq 1$ (the von Neumann condition). For this inequality to hold, $z(\kappa)$ must map the tangent to $|\kappa| = 1$ at κ_0 onto a curve that is tangent to $|z| = 1$ at z_0, as illustrated in the figure. Algebraically this is equivalent to the condition

$$\arg \frac{dz}{d\kappa}\bigg|_{\kappa_0, z_0} \equiv \arg \frac{z_0}{\kappa_0} \pmod{\pi},$$

and inserting this into (3.18) proves $C \in \mathbf{R}$, which concludes the proof.

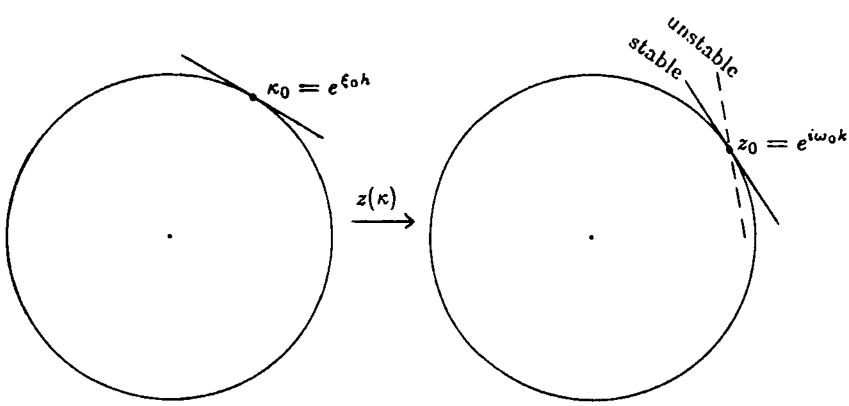

Figure 5. Analytic relationship between z and κ at a point κ_0, z_0 with $C \neq 0$.

The standard definition (3.17) of the group velocity is based on the study of the connection between perturbations $\xi_0 \to \xi_0 + \Delta\xi$ in ξ and perturbations $\omega_0 \to \omega_0 + \Delta\omega$ in ω, where it is usually supposed that both $\Delta\xi$ and $\Delta\omega$ are real, at least to first order. This amounts to studying the relationship between κ and z as κ or z is perturbed along the unit circle. However, the analyticity in the last lemma implies that one may equally well consider perturbations of κ and z off the unit circle, which is equivalent to considering complex $\Delta\xi$ and $\Delta\omega$, and this is what is done in the work of Kreiss and Osher. The following lemma makes the connection between these two approaches explicit, and in particular shows that the "perturbation test" associated with the Kreiss–Osher theory amounts to a test of group velocity.

We continue to assume that Q admits the solution (3.16) with $|\kappa_0| = |z_0| = 1$. In a sufficiently small neighborhood of z_0, the equation $P(\kappa, z) = 0$ defines ν continuous functions $\{\kappa_i(z)\}_{1 \leq i \leq \nu}$ with $\lim_{z \to z_0} \kappa_i(z) = \kappa_0$ for each i. Let Ω denote the intersection of such a neighborhood with $|z| > 1$. The von Neumann condition implies $|\kappa_i(z)| \neq 1$ for $z \in \Omega$ for each i. Therefore the following integers are well defined:

$$\nu_l\text{: number of roots } \kappa_i \quad \text{with} \quad |\kappa_i(z)| > 1 \quad \text{for} \quad z \in \Omega,$$

$$\nu_r\text{: number of roots } \kappa_i \quad \text{with} \quad |\kappa_i(z)| < 1 \quad \text{for} \quad z \in \Omega,$$

$$\nu = \nu_l + v_r.$$

The subscripts l and r stand for "leftgoing" and "rightgoing", respectively, since a solution $\kappa^j z^n$ with $|\kappa| < 1 < |z|$ can be thought of as translating to the right as n increases, and a solution with $|\kappa|, |z| > 1$, to the left. In Section 2.3 of [24] these translation speeds are made precise, and it is shown that they approach C in the limit $z \to z_0$.

LEMMA 3.3. *Let Q admit the solution* (3.16) *with* $|\kappa_0| = |z_0| = 1$, *and let* Ω, ν, ν_l, *and* ν_r *be defined as above.*

(i) *If ν is even, then $\nu_l = \nu_r = \frac{1}{2}\nu$. If ν is odd, then either $\nu_1 = \frac{1}{2}(\nu + 1)$ and $\nu_r = \frac{1}{2}(\nu - 1)$, or the reverse.*

(ii) (*Perturbation test*) *If $C \neq 0$ (so that $\nu = 1$ by Lemma* 3.2(iii), *and we can write $\kappa(z)$ for $\kappa_1(z)$), then $C < 0$ if and only if $|\kappa(z)| > 1$ for $z \in \Omega$, and $C > 0$ if and only if $|\kappa(z)| < 1$ for $z \in \Omega$. That is, C is negative if $\nu_l = 1$ and positive if $\nu_r = 1$.*

Proof: Claim (i) follows from (3.19) (cf. Theorem 9.2 of [5]). Claim (ii) follows from (3.18).

If Q admits a solution (3.16) with $|z_0| \geqq 1$ and $|\kappa_0| \neq 1$, we set $\nu_l = \nu$ and $\nu_r = 0$ if $|\kappa_0| > 1$, and $\nu_r = \nu$ and $\nu_l = 0$ if $|\kappa_0| < 1$.

We are now prepared to state some definitions that will be central to all of what follows.

DEFINITION 3.2. Let Q admit a solution

$$v_j^n = z_0^n \kappa_0^j j^\delta \tag{3.20}$$

with $|z_0| \geqq 1$ and $\delta \leqq \max\{\nu_l, \nu_r\}$.

If $|z_0| > 1$ and $|\kappa_0| < 1$ (respectively $|\kappa_0| > 1$), or if $|\kappa_0| = |z_0| = 1$ and $C > 0$ (respectively $C < 0$), then v is *strictly rightgoing* (respectively, *strictly leftgoing*).

If v is strictly rightgoing (respectively strictly leftgoing), or if $|z_0|=1$ and $|\kappa_0|<1$ (respectively $|\kappa_0|>1$), or if $|\kappa_0|=|z_0|=1$ and $C=0$ and $\delta \leqq \nu_r$ (respectively $\delta \leqq \nu_l$), then v is *rightgoing* (respectively *leftgoing*).

These definitions divide the set of solutions (3.20) with $\delta \leqq \max\{\nu_l, \nu_r\}$ into nine classes that range from strictly leftgoing modes with $|z_0|, |\kappa_0|>1$ to strictly rightgoing modes with $|z_0|>1>|\kappa_0|$. Table III summarizes this classification. Note that position (5) of the table corresponds to solutions that are both leftgoing and rightgoing.

Table III
Classification of Solutions (3.20) to Q into Leftgoing and Rightgoing Components

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
$\|z\|>1$	$\|z\|=1$	$\|z\|=1$	$\|z\|=1$	$\|z\|=1$	$\|z\|=1$	$\|z\|=1$	$\|z\|=1$	$\|z\|>1$
$\|\kappa\|>1$	$\|\kappa\|=1$	$\|\kappa\|>1$	$\|\kappa\|=1$	$\|\kappa\|=1$	$\|\kappa\|=1$	$\|\kappa\|<1$	$\|\kappa\|=1$	$\|\kappa\|<1$
	$C<0$		$C=0$	$C=0$	$C=0$		$C>0$	
			$\delta=\nu_l$ $=\nu_r+1$	$\delta\le$ $\min\{\nu_l,\nu_r\}$	$\delta=\nu_r$ $=\nu_l+1$			

strictly leftgoing: (1)–(3); strictly rightgoing: (8)–(9)

leftgoing: (1)–(5); rightgoing: (5)–(9)

Now we are finally prepared to mix modes by considering, still for fixed z, *all* of the associated values of κ. In the notation used earlier, $\{\kappa_i\}_{i=1}^{\mu}$ was the set of distinct values κ for a given z. From Lemma 3.1 it is readily seen that, according to our definitions, for all $|z|\geqq 1$ (with obvious notation)

$$\sum_{i=1}^{\mu} \nu_r^{(i)}=l, \qquad \sum_{i=1}^{\mu} \nu_l^{(i)}=r.$$

Thus *for every* $|z|\geqq 1$, *Q admits exactly l rightgoing and r leftgoing independent modes* (3.20). From this point on, let $\{\kappa_i\}_{i=1}^{l+r}$ be the set of all κ's repeated according to multiplicity, with $\{\delta_i\}$ chosen accordingly, so that $(\kappa_i, \delta_i)_{i=1}^{l}$ and $(\kappa_i, \delta_i)_{i=l+1}^{l+r}$ are the parameters associated with rightgoing and leftgoing modes, respectively. Then we can conveniently form linear combinations as follows:

DEFINITION 3.3. A *steady-state solution* to Q is a non-zero linear combination

$$v_j^n = z^n \sum_{i=1}^{l} a_i\kappa_i^j j^{\delta_i} + z^n \sum_{i=l+1}^{l+r} a_i\kappa_i^j j^{\delta_i} \tag{3.21}$$

with $|z| \geqq 1$ and complex constants $\{a_i\}$, where $\{\kappa_i\}$ and $\{\delta_i\}$ are defined in such a way that the two sums range over all right- and leftgoing modes (3.20), respectively, admitted by Q for the given z.

Equation (3.21) is nothing more than a breakdown of the solutions to Q that have regular behavior with respect to t, presented so as to emphasize the flow of energy right or left in the various component wave modes. It could be applied directly, for example, to analyze the steady-state response of a finite difference model to a sinusoidal forcing oscillation.

4. Fundamentals for the Initial Boundary Value Problem

Now we shall introduce a boundary, and specify initial, boundary, and forcing data. Since the results to come do not depend on the differential equation being approximated, but only on the difference model, we proceed directly to the numerical form of these quantities.

Consider a grid function $\{v_j^n\}$ defined for $j \geqq 0$. For $j \geqq l$, v will be determined by the finite difference formula Q of (3.4), but with a possibly non-zero forcing term F added on the right-hand side:

$$Q_{-1}v_j^{n+1} + \sum_{\sigma=0}^{s} Q_\sigma v_j^{n-\sigma} = F_j^n, \qquad l \leqq j < \infty. \tag{4.1}$$

Q (with $F=0$) is assumed to satisfy the assumptions stated in the last section. For $0 \leqq j \leqq l-1$, v will be determined by a linear set of equations with a possibly non-zero forcing term g, which can be written as follows for some integers N_1 and N_2:

$$v_j^{n+1} = \sum_{i=0}^{N_1} \sum_{\sigma=-1}^{N_2} b_{i\sigma}^{(j)} v_i^{n-\sigma} + g_j^n, \qquad b_{i,-1}^{(j)} = 0 \text{ for } i \leqq l-1, \qquad 0 \leqq j \leqq l-1. \tag{4.2}$$

(These conditions incorporate both physically meaningful boundary conditions and additional purely numerical ones.) The symbol $\bar{Q}$ will denote the difference model defined by (4.1) and (4.2). For analytical purposes we can consider solutions v_j^n defined for all $n \in \mathbf{Z}$, but in apllications v will be initialized by an initial data distribution

$$v_j^\sigma = f_j^\sigma, \qquad 0 \leqq j < \infty,\ 0 \leqq \sigma \leqq s, \tag{4.3}$$

after which (4.1) and (4.2) will be applied for $n \geqq s$. (Actually initial values are needed for $\sigma \leqq \max\{s, N_2\}$, but to avoid too much fussiness let us stick with (4.3) and assume $N_2 \leqq s$.)

Suppose F and g are identically zero. If $|z| \geqq 1$ is given, we found in the last section that Q admits an $(l+r)$-dimensional space of steady-state solutions (3.21) consisting of linear combinations of r leftgoing modes and l rightgoing ones. A solution of the form (3.21) to $\bar{Q}$, however, will have to satisfy in addition the l boundary conditions (4.2). We define in analogy to Definition 3.3:

DEFINITION 4.1. Suppose that $\bar{Q}$ with $F=g=0$ admits a non-zero solution (3.21) for some z with $|z|\geqq 1$. (That is, (3.21) satisfies (4.2) with $g=0$ as well as (4.1) with $F=0$.) Then v is a *steady-state solution* to $\bar{Q}$.

A parameter count suggests that in general, the set of steady-state solutions to $\bar{Q}$ will be a linear space of dimension $(l+r)-l=r$. To make this precise, given $|z|\geqq 1$ and hence $\{\kappa_i(z)\}$ and $\{\delta_i(z)\}$, let (3.21) be inserted in (4.2). The result is an $l\times(l+r)$-linear system of equations in $a_1,\cdots,a_{l+r}$, which can be written

$$D^{[r]}(z)a^{[r]}+D^{[l]}(z)a^{[l]}=0 \tag{4.4}$$

with $a^{[r]}=(a_1,\cdots,a_l)^T$ and $a^{[l]}=(a_{l+1},\cdots,a_{l+r})^T$, or equivalently,

$$D(z)a=0 \tag{4.4'}$$

with $a=(a_1,\cdots,a_{l+r})^T$. Here $D^{[r]}(z)$, $D^{[l]}(z)$, and $D(z)$ are matrices of dimensions $l\times l$, $l\times r$, and $l\times(l+r)$, respectively.

Equations (4.4) and (4.4′) can be interpreted as the imposition of a *reflection coefficient function* relating leftgoing waves to rightgoing ones at the boundary. If $D^{[r]}(z_0)$ is nonsingular for a given z_0, which is the typical situation, then one obtains

$$a^{[r]}=-(D^{[r]}(z_0))^{-1}D^{[l]}(z_0)a^{[l]},$$

which represents a solution to the problem of determining what waves will be reflected rightward in the steady state when a leftgoing wave is incident at the boundary (see Section 3 of [24]). On the other hand, if $D^{[r]}(z_0)$ is singular, then the reflection problem has no unique solution, for (4.4) implies that $\bar{Q}$ admits a solution (3.21) with $a^{[r]}\neq 0$, $a^{[l]}=0$.

DEFINITION 4.2. Suppose $\bar{Q}$ admits a steady-state solution (3.21) in which every mode with non-zero amplitude is rightgoing (respectively, strictly rightgoing). Then v is a *rightgoing* (respectively, *strictly rightgoing*) steady-state solution.

Our results can be summarized as follows:

LEMMA 4.1. *The function* (3.21) *is a steady-state solution to* $\bar{Q}$ *if and only if the coefficient vectors* $a^{[r]}$, $a^{[l]}$ *satisfy* (4.4). $\bar{Q}$ *admits a rightgoing steady-state solution for* $|z_0|\geqq 1$ *if and only if* $D^{[r]}(z_0)$ *is singular.*

Thus $\bar{Q}$ has a rightgoing steady-state solution with $z=z_0$ if and only if rank $(D^{[r]}(z_0))<l$. This observation suggests the following formalization of the notion of an infinite reflection coefficient:

DEFINITION 4.3. Suppose $\bar{Q}$ admits a strictly rightgoing steady-state solution for $z = z_0$, $|z_0| \geqq 1$. $\bar{Q}$ has an *infinite reflection coefficient* at $z = z_0$ if rank $(D'(z_0)) = l$, where $D'(z_0)$ is the submatrix of $D(z_0)$ containing only those columns corresponding to strictly rightgoing or strictly leftgoing components.

Now we consider various definitions of stability for $\bar{Q}$. First, let $\|\cdot\|_+$ denote the norm defined by

$$\|\phi\|_+^2 = h \sum_{j=0}^{\infty} |\phi_j|^2, \tag{4.5}$$

(cf. (3.6)), and let l_+^2 denote the set of spatial grid functions ϕ on $j \geqq 0$ for which this number is finite. The simplest stability definition has to do with dependence of interior solution values on initial data. Let F_j^n and g_j^n be identically zero again. Let S be the solution operator $S: (l_+^2)^{s+1} \to (l_+^2)^{s+1}$ defined by

$$S: (v^n, \cdots, v^{n-s}) \mapsto (v^{n+1}, \cdots, v^{n-s+1}). \tag{4.6}$$

Let these $(s+1)$-level vectors be normed in the natural way by

$$\|(v^n, \cdots, v^{n-s})\|^2 = \sum_{\sigma=0}^{s} \|v^{n-\sigma}\|_+^2,$$

and let $\|S\|$ be the induced operator norm applied to S. We define:

DEFINITION 4.4. $\bar{Q}$ is *l^2-stable* if there exists a constant M such that

$$\|S^n\| \leqq M \quad \text{for all} \quad n \geqq 0. \tag{4.7}$$

Another notion of stability concerns dependence of interior solution values on boundary data. This time, assume that F is identically zero and that g_j^n is zero for $n < s$ but possibly non-zero for $n \geqq s$. Let v_j^n be computed for $n \geqq 0$ with homogeneous initial data $f = 0$ in (4.3). For each $n \geqq s+1$, let $S_{bc}^{(n)}$ denote the operator

$$S_{bc}^{(n)}: g \mapsto v^n$$

with norm induced by the l_+^2 norm (4.5) on v^n and a similar norm $\|\cdot\|_l$ on g, defined by

$$\|g\|_l^2 = h \sum_{j=0}^{l-1} \sum_{n=s}^{\infty} |g_j^n|^2. \tag{4.8}$$

DEFINITION 4.5. $\bar{Q}$ is *l^2-stable with respect to boundary data* if there exists a constant M such that

$$\|S_{bc}^{(n)}\| \leqq M \quad \text{for all} \quad n \geqq 0. \tag{4.9}$$

It can readily be seen that although l^2-stability does not imply l^2-stability with respect to boundary data, the gap between the two is at most a factor of $\sqrt{n}$:

LEMMA 4.2. *For any $\bar{Q}$ and $n \geqq 1$,*

$$\|S_{bc}^{(n)}\| \leqq \sqrt{n} \max_{0 \leqq \sigma \leqq n-1} \|S^\sigma\|. \tag{4.10}$$

Proof: We apply the discrete form of Duhamel's principle. Given boundary data $g = \{g_j^n\}$, define an associated family of initial data distributions $w^n \in (l_+^2)^{s+1}$ by

$$w^n = (\hat{g}^{s+n}, 0, \cdots, 0), \qquad n \geqq 0,$$

where $\hat{g}^n \in l_+^2$ is simply

$$\hat{g}^n = (g_0^n, \cdots, g_{l-1}^n, 0, 0, \cdots).$$

Then by (4.1)–(4.3) one has

$$(v^s, \cdots, v^0) = (0, \cdots, 0),$$

$$(v^{1+s}, \cdots, v^1) = (\hat{g}^2, 0, \cdots, 0) = w^0,$$

$$(v^{2+s}, \cdots, v^2) = Sw^0 + w^1,$$

and after n steps,

$$(v^{n+s}, \cdots, v^n) = S^{n-1}w^0 + S^{n-2}w^1 + \cdots + w^{n-1}. \tag{4.11}$$

If we set

$$\mu = \max_{0 \leqq \sigma \leqq n-1} \|S^\sigma\|,$$

then (4.11) implies

$$\|(v^{n+s}, \cdots, v^n)\| \leqq \mu(\|w^0\| + \cdots + \|w^{n-1}\|);$$

hence

$$\|v^n\|_+ \leqq \mu(\|w^0\| + \cdots + \|w^{n-1}\|),$$

and therefore, by the Hölder inequality,

$$\|v^n\|_+^2 \leqq \mu^2 n(\|w^0\|^2 + \cdots + \|w^{n-1}\|^2)$$
$$= \mu^2 n(\|\hat{g}^s\|_+^2 + \cdots + \|\hat{g}^{n+s-1}\|_+^2)$$
$$\leqq \mu^2 n \|g\|_t^2.$$

Thus $\|v^n\|_+ \leqq \mu\sqrt{n}\|g\|_t$, and since g was arbitrary, this establishes (4.10).

The final definition of stability that we shall consider comes from the paper of Gustafsson, Kreiss, and Sundström [5], where it is given as Definition 3.3. This time, let f_j be zero and consider functions F_j^n and g_j^n that are zero for $n < s$ but possibly non-zero for $n \geqq s$. For a distribution ϕ_j^n indexed by $0 \leqq j < \infty$, $n \geqq 0$, define the norm $\|\phi\|_{x,t}$ by

$$\|\phi\|_{x,t}^2 = hk \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} |\phi_j^n|^2.$$

DEFINITION 4.6. $\bar{Q}$ is *GKS-stable* if there exist constants $\alpha_0 \geqq 0$ and M such that, for all $\alpha > \alpha_0$, the following estimate holds for all sufficiently small k:

$$\left(\frac{\alpha-\alpha_0}{1+\alpha k}\right)^2 \|e^{-\alpha t} v\|_{x,t}^2 + \left(\frac{\alpha-\alpha_0}{1+\alpha k}\right) \|e^{-\alpha t} v\|_t^2 \leqq M\left[\|e^{-\alpha(t+k)} F\|_{x,t}^2 + \left(\frac{\alpha-\alpha_0}{1+\alpha k}\right)\|e^{-\alpha(t+k)} g\|_t^2\right]. \tag{4.12}$$

Remarks on the GKS stability definition. It is not obvious at first inspection what the function is of the various terms in the GKS definition of stability. In fact, (4.12) has two principal features that set it apart from l^2-stability with respect to initial and boundary data:

1. The second term on the left in (4.12) requires that not only interior values but also boundary values depend stably on the forcing data. This is an essential restriction, which becomes significant whenever $\bar{Q}$ admits rightgoing but not strictly rightgoing steady-state solutions. In particular it gives rise to Theorem 1b. Sometimes it is important to have such an estimate – for example, in a problem where physical quantities at the boundary are of independent interest, or in any situation where boundary values from one computation will be used as forcing data in another (Section 5.4 of [24]).

2. All of the norms in (4.12) are integrals over t rather than estimates at fixed t. (The decay factors $e^{-\alpha t}$, the normalizing terms $(\alpha-\alpha_0)/(1+\alpha k)$, and the replacement of f by F all stem from this.) This is a technical restriction made necessary by the proof of the main stability theorem (our Theorem 1a) in [5], which depends upon Fourier transforms with respect to t. For theoretical applications the reliance on integrals in t is an unfortunate weakness in the GKS theory, but in practice it seems to have no significance. Indeed it appears that GKS-stability may imply l^2-stability, although up to now this has not been proved.

Because of these complications, the application of the concept of GKS-stability – for example, to reach conclusions regarding convergence (see [4]) – is often not an easy matter. However, there are two important advantages of the GKS definition, which motivated its introduction in [5]:

1. This is the only stability definition for which necessary and sufficient conditions have been obtained for difference models of general form.

2. It can be shown that if a difference model $\bar{Q}$ is GKS-stable, then the same holds for nearby difference models defined by coefficients sufficiently close to those of $\bar{Q}$. In other words, the set of GKS-stable difference models is *open*. On the basis of this fact it can be shown that perturbations introduced by smoothly variable coefficients, lower order terms, and multiple boundaries preserve GKS-stability (see [5], [9], [11]). Thus GKS-stability is robust in a way that l^2 and other notions of stability are not.

5. Growth Rate Results

We are now in a position to prove Theorems 1a–6a of Table I.

THEOREM 1a. *$\bar{Q}$ is GKS-stable if and only if it admits no rightgoing steady-state solutions.*

Proof: This is a restatement of the result that is generally considered the main theorem of [5], where it appears as Lemma 10.3 and the sentence following in the form of a determinant condition: $\bar{Q}$ is GKS-stable if and only if det $D^{[r]}(z) \neq 0$ for all z with $|z| \geqq 1$, where $D^{[r]}$ is the $l \times l$-matrix of (4.4). (See [5] for certain restrictive assumptions on the range of permissible models $\bar{Q}$.) By Lemma 4.1, the determinant condition is equivalent to the condition that there are no rightgoing steady-state solutions.

The rest of our proofs will be limited to the case of two-level ($s = 0$) difference models. They are all based on Lemma 5.1 below, which states that if a wave packet is smooth, then it propagates approximately at speed C. Proving this involves estimating a Fourier integral that has a standard form: if one divides through by the carrier oscillation $e^{i(\omega_0 t + \xi_0 x)}$, then what remains is the same kind of integral as governs the propagation of a smooth signal under a consistent finite difference model of the equation $u_t = -Cu_x$, where $C = C(\xi_0, \omega_0)$. A variety of estimates for such integrals are available in the literature on stability of difference schemes in the maximum norm (cf. [1]). However, all we need is a very special case. Therefore, rather than appeal to existing theorems, which would introduce undetermined constants and obscure the simplicity of what is going on, we give the following argument from first principles.

Let h and k be fixed and let Q be a two-level constant-coefficient Cauchy stable difference formula that admits a solution $\exp\{i(\omega_0 t + \xi_o x)\}$ with $\xi_0, \omega_0 \in \mathbf{R}$. By Lemma 3.2, there exists $C \in \mathbf{R}$ such that the dispersion function $\omega(\xi)$ satisfies

$$(5.1) \qquad \left.\begin{aligned} \omega(\xi) &= \omega_0 - C(\xi - \xi_0) + \tau(\xi) \\ |\tau(\xi)| &\leqq M(\xi - \xi_0)^2 \end{aligned}\right\} \quad \text{for all} \quad z \in \mathbf{R},$$

for some constant M. By Cauchy stability, we have $\mathcal{I}m\ \omega \geqq 0$ for all $\xi \in \mathbf{R}$, which implies $\mathcal{I}m\ \tau(\xi) \geqq 0$ also. Since $\eta \mapsto e^{i\eta}$ is a contraction map for $\mathcal{I}m\ \eta \geqq 0$, this implies

$$|e^{i\omega(\xi)t}-\exp\{i(\omega_0-C(\xi-\xi_0))t\}|\leqq t|\tau(\xi)|\leqq Mt(\xi-\xi_0)^2 \tag{5.2}$$

for any $t\geqq 0$.

In what follows the Fourier transform and its inverse are defined by

$$\hat{f}(\xi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\xi x}f(x)\,dx,\qquad f(x)=\int_{-\infty}^{\infty}e^{i\xi x}\hat{f}(\xi)\,d\xi. \tag{5.3}$$

LEMMA 5.1. *Let $p(x)$ belong to C_0^2 (twice continuously differentiable with compact support) and satisfy $\widehat{p''}\in L^1$. Let Q be applied (for all $x\in\mathbf{R}$) with initial data*

$$v^0(x)=\exp\{i\xi_0x\}p(x).$$

Then, for any $n\geqq 0$ and any $x\in\mathbf{R}$, $v^n(x)$ satisfies

$$|v^n(x)-\exp\{i(\omega_0t+\xi_0x)\}p(x-Ct)|\leqq Mt\|\widehat{p''}\|_1, \tag{5.4}$$

where $t=nk$, $\|\cdot\|_1$ is the L^1 norm, and M is the constant of (5.1).

Proof: Obviously $p\in L^2$; hence $v^0\in L^2$ also, and we can use Fourier transforms. We get

$$\begin{aligned} v^n(x)&=\int e^{i(\omega(\xi)t+\xi x)}\,\widehat{v^0}(\xi)\,d\xi\\ &=\int e^{i(\omega(\xi)t+\xi x)}\,\hat{p}(\xi-\xi_0)\,d\xi\\ &=\int [e^{i\omega(\xi)t}+\exp\{i(\omega_0-C(\xi-\xi_0))t\}\\ &\qquad -\exp\{i(\omega_0-C(\xi-\xi_0))t\}]\,e^{i\xi x}\hat{p}(\xi-\xi_0)\,d\xi. \end{aligned}$$

The integral involving the middle term in the brackets is just

$$\begin{aligned} \exp\{i(\omega_0t+\xi_0x)\}&\int \exp\{i(\xi-\xi_0)(x-Ct)\}\hat{p}(\xi-\xi_0)\,d\xi\\ &=\exp\{i(\omega_0t+\xi_0x)\}\,p(x-Ct). \end{aligned}$$

Thus we have, using (5.2),

$$\begin{aligned} &|v^n(x)-\exp\{i(\omega_0t+\xi_0x)\}\,p(x-Ct)|\\ &\quad=\left|\int (e^{i\omega(\xi)t}-\exp\{i(\omega_0-C(\xi-\xi_0))t\})\,e^{i\xi x}\hat{p}(\xi-\xi_0)\,d\xi\right|\\ &\quad\leqq\int Mt(\xi-\xi_0)^2|\hat{p}(\xi-\xi_0)|\,d\xi \end{aligned}$$

$$= Mt \int |\xi^2 \hat{p}(\xi)|\, d\xi$$

$$= Mt \int |\widehat{p''}(\xi)|\, d\xi = Mt \|\widehat{p''}\|_1,$$

concluding the proof.

If p is smooth, then the right-hand side of (5.4) is small. To make p smooth we broaden it, while continuing to hold h and k fixed, although the same results could be obtained by leaving p fixed and reducing h and k.

LEMMA 5.2. *Suppose $p(x) = P(\varepsilon x)$ for some fixed $P \in C_0^2$ with $\widehat{P''} \in L^1$. Then*

$$\|\widehat{p''}\|_1 = \varepsilon^2 \|\widehat{P''}\|_1 .$$

Proof: The verification is straightforward and is given in Appendix B of [24].

Here is the first set of growth rate theorems.

THEOREM 2a. *Suppose $\bar{Q}$ admits a strictly rightgoing steady-state solution. Then it is l^2-unstable, and satisfies*

$$\|S^n\| \geqq \text{const.} \sqrt{n}$$

for infinitely many integers $n > 0$.

THEOREM 5a. *Suppose $\bar{Q}$ admits a strictly rightgoing steady-state solution. Then it is also l^2-unstable with respect to boundary data, and satisfies*

$$\|S_{bc}^{(n)}\| \geqq \text{const. } n \tag{5.5}$$

for all $n > 0$.

THEOREM 6a. *In particular,* (5.5) *holds if $\bar{Q}$ admits a strictly rightgoing steady-state solution with infinite reflection coefficient.*

Proofs: Theorem 6a is a special case of Theorem 5a, and we state it separately only for the sake of Table I. Theorem 2a follows from Theorem 5a by Lemma 4.2 (equation (4.10)). Thus our problem is to prove Theorem 5a.

First we observe that if $\bar{Q}$ admits a strictly rightgoing steady-state solution with $|z_0| > 1$ (i.e., of Godunov–Ryabenkii type [20]), then catastrophic growth at the rate $|z_0|^n$ will take place, which is much more rapid than (5.5). Assume therefore that $\bar{Q}$ admits a strictly rightgoing stady-state solution (cf. (3.21))

$$z_0^n \sum_{i=1}^{l} a_i \kappa_i^j = \exp\{i\omega_0 t\} \sum_{i=1}^{l} a_i \exp\{i\xi_i x\} \qquad t = nk,\ x = jh \tag{5.6}$$

with $|z_0| = 1$. The hypothesis that (5.6) is strictly rightgoing implies that each κ_i with $a_i \neq 0$ is simple (Lemma 3.2(iii)) and has $|\kappa_i| = 1$, i.e., $\xi_i \in \mathbf{R}$, with an associated group velocity $C_i = C(\kappa_i, z_0) > 0$.

Here is the idea of the proof: Let h, k be fixed, and consider the *Cauchy problem modeled by* Q – that is, ignore the boundary to begin with. Construct a wave packet consisting of a smooth envelope multiplied by (5.6). Initially the packet will have width N and lie to the left of $x=0$. But by time $t=Nk$, much of the energy will have traveled into $x>0$, as illustrated in Figure 6.

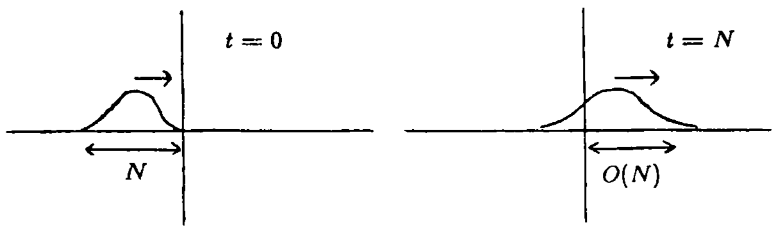

Figure 6. "Equivalent boundary data" construction for the proof of Theorem 6a.

Now the key point is this: *the solution* $\{v_j^n\}$ *obtained under* Q *in this manner is identical in* $j\geqq 0$ *to the solution* $\{\bar v_j^n\}$ *obtained if* $\bar Q$ *is applied with initial data zero and boundary data equal to the numbers* $\{g_j^n\}=\{[G(v)]_j^n\}$ *produced when* $\{v_j^n\}$ *is inserted into* (4.2):

$$[G(v)]_j^n = v_j^{n+1} - \sum_{i=0}^{N_1}\sum_{\sigma=-1}^{N_2} b_{i\sigma}^{(j)} v_i^{n-\sigma}. \tag{5.7}$$

In other words, $\{v_j^n\}$ would be an exact solution of $\bar Q$ with $g=0$, as well as of Q, if it happened to satisfy the homogeneous boundary conditions. It does not, but it does satisfy $\bar Q$ when we take just the right inhomogeneous "equivalent boundary data" $g=G(v)\neq 0$. Since v^0 is made up of a steady-state solution to $\bar Q$ multiplied by a smooth envelope, the homogeneous boundary conditions are nearly satisfied, so $G(v)$ will be small. In fact we shall pick v^0 such that $\|v^0\|_+=0$ and

$$\|v^N\|_+ \geqq \text{const.}\sqrt{N}, \tag{5.8}$$

but such that $G(v)$ satisfies

$$\|G(v)\|_t \leqq \text{const.}/\sqrt{N}. \tag{5.9}$$

(These norms were defined in (4.5) and (4.8).) Since $\|S_{bc}^{(n)}\|=\sup_g\|v^n\|_+/\|g\|_t$, (5.8) and (5.9) imply (5.5).

Here are the details. Let $P\in C_0^2$ be a fixed function with $P(x)>0$ on $(-1,0)$, $P(x)=0$ elsewhere, and $\widehat{P''}\in L^1$, and write $P'_{\max}=\sup|P'(x)|$. Let N be given, and set $T=Nk$. Consider the Cauchy problem for Q (for all $x\in \mathbf{R}$) with initial data

$$v^0(x)=\sum_{i=1}^{l} a_i \exp\{i\xi_i x\}p_i(x), \quad p_i(x)=P(x/C_iT). \tag{5.10}$$

Let M_i be the constant of Lemma 5.1 for the wave ξ_i, ω_0. For any n write $t = nk$. By Lemmas 5.1 and 5.2, we then have

$$\left| v^n(x) - \sum a_i \exp\{i(\omega_0 t + \xi_i x)\} p_i(x - C_i t) \right| \leq t \sum |a_i| M_i \|\widehat{p_i''}\|_1 = \frac{t\|\widehat{P''}\|}{T^2} \sum |a_i| M_i C_i^{-2}.$$

In particular, for $n \leqq N$ and hence $t \leqq T$, this equation together with (5.10) implies

$$\left| v^n(x) - \sum a_i \exp\{i(\omega_0 t + \xi_i x)\} P\left(\frac{x}{C_i T} - \frac{t}{T}\right) \right| \leqq \frac{A_1}{N}, \tag{5.11}$$

where $A_1 = \|\widehat{P''}\|_1 \sum |a_i| M_i / kC_i^2$.

Now we are equipped to show that $\|g\|_t$ is small. Given $n \leqq N$ and $t = nk$, define for all σ and j

$$\tilde{v}^{n-\sigma}(x) = P\left(\frac{-t}{T}\right) \sum a_i \exp\{i(\omega_0(t - \sigma k) + \xi_i x)\}.$$

Then we have

$$\begin{aligned} |v^{n-\sigma}(x) - \tilde{v}^{n-\sigma}(x)| \leqq & \left| v^{n-\sigma}(x) - \sum a_i \exp\{i(\omega_0(t-\sigma k) + \xi_i x)\} P\left(\frac{x}{C_i T} - \frac{t - \sigma k}{T}\right) \right| \\ & + \sum |a_i \exp\{i(\omega_0(t-\sigma k) + \xi_i x)\}| \left| P\left(\frac{x}{C_i T} - \frac{t-\sigma k}{T}\right) - P\left(\frac{-t}{T}\right) \right| \\ \leqq & \frac{A_1}{N} + \sum |a_i \exp\{i(\omega_0(t-\sigma k) + \xi_i x)\}| P'_{\max} \left| \frac{x}{C_i T} + \frac{\sigma k}{T} \right|. \end{aligned}$$

Therefore, for some $A_2 < \infty$, if $x = jh$ with j restricted to a bounded range, we have

$$|v_j^{n-\sigma} - \tilde{v}_j^{n-\sigma}| \leqq \frac{A_2}{N}, \qquad 0 \leqq j \leqq N_1, \ -1 \leqq \sigma \leqq N_2. \tag{5.12}$$

Now by definition, $\tilde{v}$ is the steady-state solution (5.6) multiplied by the constant $P(-t/T)$, which implies

$$G(\tilde{v}) = 0.$$

Consequently we have from (5.7)

$$|G(v)_j^n| = |G(v - \tilde{v})_j^n| \leqq |v_j^{n+1} - \tilde{v}_j^{n+1}| + \sum_{i=0}^{N_1} \sum_{\sigma=-1}^{N_2} |b_{i\sigma}^{(j)}| \, |v_i^{n-\sigma} - \tilde{v}_i^{n-\sigma}|.$$

By (5.12), each summand on the right is $O(1/N)$. Therefore,

$$|G(v)_j^n| \leqq \frac{A_3}{N}, \qquad 0 \leqq j \leqq l-1, \ 0 \leqq n \leqq N-1, \tag{5.13}$$

for some A_3. Hence

$$\|G(v)\|_t^2 \leqq h \sum_{j=0}^{l-1} \sum_{n=0}^{N-1} \frac{A_3^2}{N^2} = \frac{hlA_3^2}{N}$$

(to be precise, this bound holds for a function g which is equal to $G(v)$ for $n \leqq N-1$ and to zero thereafter), and taking the square root gives (5.9).

The other half of the argument is to show that $\|v^N\|_+$ (equation (4.5)) is big. Now by definition of the numbers κ_i, the steady-state solution (5.6) cannot be zero at more than $l-1$ consecutive grid points without being identically zero. It follows that one has

$$\sum_{j=0}^{\infty} \left| \sum_{i=1}^{l} a_i \exp\{i(\omega_0(t-\sigma k)+\xi_i x)\} P\left(\frac{jh}{C_i T}-1\right)\right|^2 > A_4 N$$

for some A_4, so long as $T \geqq T_0 \geqq lh/\max_i C_i$. Taking the square root and using (5.11) we arrive at (5.8), as desired.

Now we add the hypothesis of an infinite reflection coefficient.

THEOREM 3a. *Suppose $\bar{Q}$ admits a strictly rightgoing steady-state solution with infinite reflection coefficient* (*Definitions* 4.1–4.3). *Then*

$$\|S^n\| \geqq \text{const. } n$$

for infinitely many $n > 0$.

Proof: Once again the case of a steady-state solution with $|z_0| > 1$ is easy; so assume that $\bar{Q}$ admits a strictly rightgoing steady-state solution (5.6) with $|z_0| = |\exp\{i\omega_0\kappa\}| = 1$, hence also $\xi_i \in \mathbf{R}$ and $C_i > 0$ for each i such that $a_i \neq 0$. Our argument will make use only of these and other components ξ_i with $\xi_i \in \mathbf{R}$ and $C_i \neq 0$ (positive or negative). Rather than introduce new notation in place of l, r, a_i, etc., let us assume for simplicity that all components ξ_i, $1 \leqq i \leqq l+r$, are of this kind. In this situation Definition 4.3 for an infinite reflection coefficient reduces to the condition rank $D(z_0) = l$ (cf. equation (4.4′)).

The proof is an extension of that of Theorem 5a, which related the given initial boundary value problem for $\bar{Q}$ to a Cauchy problem for Q. For Theorem 5a, we constructed initial data v^0 with $\|v^0\|_+ = 0$, $\|v^N\|_+ \geqq \text{const.} \sqrt{N}$, and $\|G(v)\|_t \leqq \text{const.}/\sqrt{N}$, where $G(v)$ was the "equivalent boundary data" distribution (5.7) for v. Taking the ratio of norms we then established (5.5). This time, we shall add a small essentially leftgoing component to v so as to obtain a solution w that satisfies the boundary conditions of $\bar{Q}$ to a higher order, although $\|w^0\|_+$ will no longer be zero. To be precise, we shall construct w^0 so that

$$\|w^n - v^n\| \leqq \text{const.}/\sqrt{N}, \qquad 0 \leqq n \leqq N, \tag{5.14}$$

but

$$\|G(w)\|_t \leqq \text{const.}/N. \tag{5.15}$$

Since $\|v^0\|_+ = 0$, (5.14) implies

$$\|w^0\|_+ \leqq \text{const.}/\sqrt{N}, \tag{5.16}$$

and together with (5.8), it also gives

$$\|w^N\|_+ \geqq \text{const.}\sqrt{N}, \tag{5.17}$$

assuming N is sufficiently large. Equations (5.15)–(5.17) imply that for each N, either $\|S^N\| \geqq \text{const.}\, N$ or $\|S_{bc}^{(N)}\| \geqq \text{const.}\, N^{3/2}$. By Lemma 4.2, these facts imply $\|S^N\| \geqq \text{const.}\, N$ for infinitely many N.

Thus the proof comes down to finding w^0 such that (5.14) and (5.15) hold. Here is the construction, which is based on Fourier transform manipulations.

First, let N and $T = Nk$ be given and let P and p_i be defined as before (equation (5.10)). Since P has compact support, $\hat{P}$ is entire. Assume further that it satisfies

$$|\hat{P}(\xi)| \leqq \text{const.}\, |\xi|^{-3} \tag{5.18}$$

for ξ in any fixed strip $|\mathcal{I}m\, \xi| \leqq \gamma$. (For example, take $P(x) = \phi(6x+3)$, where ϕ is the threefold convolution $\chi_{[-1,1]} * \chi_{[-1,1]} * \chi_{[-1,1]}$, whose transform is $\hat{\phi}(\xi) = [\sin(\xi)/\pi\xi]^3$.)

By Lemma 3.2, $\{\xi_i\}$ are the point values at $\omega = \omega_0$ of $l+r$ functions $\xi_i(\omega)$ that are analytic and invertible in some neighborhood of ω_0. Define $I = [\omega_0 - 1/\sqrt{N}, \omega_0 + 1/\sqrt{N}]$. For all sufficiently large N, I lies in such a neighborhood. Now since (5.6) is a rightgoing steady-state solution, Lemma 4.1 implies $D(\omega_0)a = 0$, where $a = (a_1, \cdots, a_{l+r})^T = (a_1, \cdots, a_l, 0, \cdots, 0)^T$ and $D(\omega)$ is the $l \times (l+r)$ matrix of (4.4′), which for convenience we now view as a function of ω instead of z. By the assumption of an infinite reflection coefficient, as discussed above, $D(\omega_0)$ has full rank l. Therefore, by the implicit function theorem, there exists a coefficient-vector function $A(\omega)$, analytic on I for all sufficiently large N, that satisfies

$$D(\omega)A(\omega) = 0 \quad \text{for all} \quad \omega \in I \tag{5.19}$$

and

$$A(\omega) - a = O(\omega - \omega_0). \tag{5.20}$$

Assuming N is sufficiently large, define now

$$W^n(x) = \sum_{i=1}^{l+r} \int_I A_i(\omega) \exp\{i(\omega t + \xi_i(\omega)x)\} T\hat{P}((\omega-\omega_0)T)\, d\omega, \qquad t = nk. \tag{5.21}$$

W^n is an integral of functions that satisfy both Q and $\bar{Q}$, and so W^n must satisfy these also. The fact that it satisfies $\bar{Q}$ can be expressed in the notation of (5.7) as

$$G(W)=0. \tag{5.22}$$

In general W^n does not belong to L^2, but grows exponentially as $x \to \pm\infty$, because the wave numbers $\xi_i(\omega)$ need not be real for $\omega \in I$ except at $\omega = \omega_0$.

Let S_i denote $\xi_i(I)$. Cauchy stability (see Figure 5 in Section 3) implies that S_i is a smooth arc lying tangent to the real axis at ξ_i, with $S_1, \cdots, S_l$ situated above or on the axis and $S_{l+1}, \cdots, S_{l+r}$ below or on it, as indicated in Figure 7.

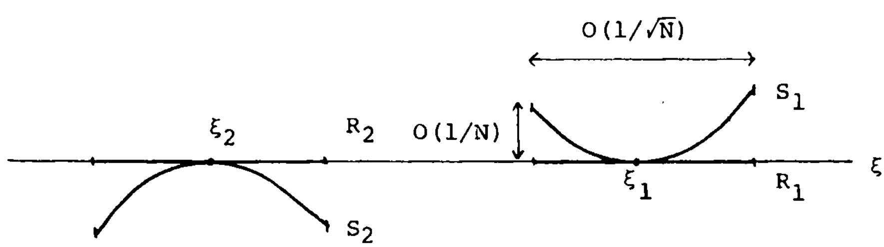

Figure 7. Integration contours in the proof of Theorem 3a for a model with $l=r=1$.

(If Q is strictly nondissipative, each S_i lies in **R**, and the proof can be simplified.) Since I and S_i are related analytically, (5.21) can be transplanted to an equivalent integral over $\bigcup_{i=1}^{l+r} S_i$:

$$W^n(x)=\sum \int_{S_i} A_i(\omega(\xi))\, e^{i(\omega(\xi)t+\xi x)} T\hat{P}((\omega(\xi)-\omega_0)T)\frac{d\omega(\xi)}{d\xi}\, d\xi. \tag{5.23}$$

To obtain a function in L^2, we replace each S_i by the nearby contour $R_i = \mathcal{Re}\, S_i$, as shown in Figure 7, and define

$$w^n(x)=\sum \int_{R_i} A_i(\omega(\xi))\, e^{i(\omega(\xi)t+\xi x)} T\hat{P}((\omega(\xi)-\omega_0)T)\frac{d\omega(\xi)}{d\xi}\, d\xi. \tag{5.24}$$

For sufficiently large N, R_i and S_i both lie in the domains of analyticity of $\omega(\cdot)$ and $A_i(\omega(\cdot))$; so (5.24) makes sense. It is also clear that w^n as defined satisfies Q. What remains is to establish (5.14) and (5.15).

To prove (5.15), observe that since the integrand in (5.23) and (5.24) is analytic, $W^n(x)-w^n(x)$ can be written as

$$W^n(x)-w^n(x)=\sum \int_{D_i} A_i(\omega(\xi))\, e^{i(\omega(\xi)t+\xi x)} T\hat{P}((\omega(\xi)-\omega_0)T)\frac{d\omega(\xi)}{d\xi}\, d\xi, \tag{5.25}$$

where D_i consists of the two vertical segments connecting $\xi_i(\omega_0 \pm 1/\sqrt{N})$ and $\mathcal{Re}\,[\xi_i(\omega_0 \pm 1/\sqrt{N})]$. Each of these segments has length $O(1/N)$ and lies at a distance at least const. $/\sqrt{N}$ from ξ_i, where by (5.18), $\hat{P}((\omega(\xi)-\omega_0)T)$ has magnitude $O(N^{-3/2})$. The terms A_i and $d\omega/d\xi$ in (5.25) are both uniformly bounded on D_i independently of N, and for $x=jh$ and $t=nk$ with $|j|\leqq J$ (any fixed constant) and $n\leqq N$, the same is true of $e^{i(\omega(\xi)t+\xi x)}$. Therefore (5.25) implies

$$|W_j^n - w_j^n| \leqq \text{const.}\,\frac{1}{N} N N^{-3/2} = O(N^{-3/2})$$

for $|j|\leqq J$, $n\leqq N$, hence $G(W-w)_j^n = O(N^{-3/2})$, and hence by (5.22) also

$$G(w)_j^n = O(N^{-3/2}).$$

Equation (5.15) follows from this just as (5.9) followed from (5.13).

To prove (5.14), we subtract from (5.24) the Fourier integral for $v^n(x)$,

$$\begin{aligned} v^n(x) &= \int_{\mathbf{R}} e^{i(\omega(\xi)t+\xi x)}\widehat{v_0}(\xi)\,d\xi \\ &= \sum \int_{\mathbf{R}} a_i C_i\, e^{i(\omega(\xi)t+\xi x)} T\hat{P}((\xi-\xi_i)C_i T)\,d\xi, \end{aligned} \tag{5.26}$$

(see (5.10)), and break up the result into four pieces

$$w^n(x) - v^n(x) = I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = -\sum \int_{\mathbf{R}-R_i} a_i C_i\, e^{i(\omega(\xi)t+\xi x)} T\hat{P}((\xi-\xi_i)C_i T)\,d\xi,$$

$$I_2 = \sum \int_{R_i} [A_i(\omega(\xi)) - a_i] C_i\, e^{i(\omega(\xi)t+\xi x)} T\hat{P}((\xi-\xi_i)C_i T)\,d\xi,$$

$$I_3 = \sum \int_{R_i} A_i(\omega(\xi)) \left[\frac{d\omega(\xi)}{d\xi} - C_i\right] e^{i(\omega(\xi)t+\xi x)} T\hat{P}((\xi-\xi_i)C_i T)\,d\xi,$$

$$I_4 = \sum \int_{R_i} A_i(\omega(\xi))\, e^{i(\omega(\xi)t+\xi x)} T[\hat{P}((\omega(\xi)-\omega_0)T) - \hat{P}((\xi-\xi_i)C_i T)] \frac{d\omega(\xi)}{d\xi}\,d\xi.$$

To obtain (5.14) it is enough to show that $\|I_j\| \leqq \text{const.}/\sqrt{N}$ for each j, that is,

$$\|I_j\|^2 \leqq \text{const.}/N \quad \text{for} \quad j=1,2,3,4, \qquad n\leqq N. \tag{5.27}$$

Since each I_j is a Fourier integral $\int e^{i\xi x} F_j(\xi)\,d\xi$, Parseval's theorem implies that (5.27) is equivalent to the bound

$$\int |F_j(\xi)|^2\,d\xi \leqq \text{const.}/N \tag{5.28}$$

on the norm of the Fourier integrand. For $j=1$, (5.28) holds because $\hat{P}((\xi-\xi_i)C_iT))$ is small for $\xi \in \mathbf{R}-R_i$: by (5.18) one has an order of magnitude

$$\|I_1\|^2 \approx \sum \int_{1/\sqrt{N}}^{\infty} [N(\xi N)^{-3}]^2 \, d\xi \approx N^{-4}\xi^{-5}\Big|_{1/\sqrt{N}}^{\infty} \approx N^{-3/2}.$$

(The approximation sign indicates equality up to order in N, ignoring constants.) For $j=2,3,4$, on the other hand, (5.28) holds because the integrand contains a term (the one written in square brackets) that is 0 at $\xi=\xi_i$ and grows at most linearly (and independently of N) with $\xi-\xi_i$. One estimates

$$\|I_{2,3,4}\| \approx \sum \int_{-1/\sqrt{N}}^{1/\sqrt{N}} [\xi N \hat{P}(\xi N)]^2 \, d\xi \approx \int_{-1/N}^{1/N} [\xi N]^2 \, d\xi \approx N^{-1}.$$

Together these bounds establish (5.27).

6. Sharpness Results

In this section we show that Theorems 1a–6a of Table I are sharp in the following sense: there exist difference models satisfying the hypotheses of each theorem for which the unstable growth rate is no greater than that asserted. The proofs consist of constructing explicit examples of such models. The desired growth rate bounds are established in each case by the following trick, which is related to the proof of Theorem 3a: the given initial boundary value problem is imbedded by a clever extension of the initial data to $j<0$ in an initial value problem whose solution satisfies the original boundary condition exactly. Such a trick cannot of course be accomplished for arbitrary difference models, and in particular the method is limited to problems in one space dimension.

Our first result, together with Theorem 2a, shows that the distinction between rightgoing and strictly rightgoing steady-state solutions has a real effect on l^2-stability.

THEOREM 1b. *GKS-instability does not imply l^2-instability. That is, there exists a model $\bar{Q}$ admitting a rightgoing steady-state solution for which*

$$\|S^n\| \leqq \text{const.} \quad \textit{for all} \quad n \geqq 1. \tag{6.1}$$

Proof: A simple proof of such a claim can be based on the "transparent interface anomaly" (cf. [20], [21]) in which a Cauchy stable model of $u_t = u_x$ on $-\infty < j < \infty$ is folded into an equivalent initial boundary value problem model in two variables on $j \geqq 0$. If the original formula is leap frog, Crank–Nicolson, or any other Cauchy stable formula that admits a wave solution with $C=0$, then the folded model $\bar{Q}$ is GKS-unstable but l^2-stable.

However, since the definitions here have been limited to scalar problems, we must give a proof involving just a single variable. Consider again the leap frog formula from (2.1),

$$Q\colon\ v_j^{n+1}=v_j^{n-1}+\lambda(v_{j+1}^n-v_{j-1}^n), \tag{6.2}$$

with $\lambda<1$. (Crank–Nicolson could also be used here.) For convenience let the domain be $j\geqq -1$ instead of $j\geqq 0$, redefining $\|\cdot\|_+$ and l_+^2 correspondingly, and take the numerical boundary condition to be

$$v_{-1}^{n+1}=-v_1^{n+1}. \tag{6.3}$$

The model $\bar{Q}$ defined by (6.2), (6.3) admits rightgoing steady-state solutions $v_j^n=z^n\kappa^j$ with $\kappa=\pm i$, so by Theorem 1a it is GKS-unstable. But by (2.4) these solutions have $C=0$ and thus are not strictly rightgoing.

We establish (6.1) as follows. Given arbitrary initial data v^0, $v^1\in l_+^2$ satisfying (6.3), let v^0 and v^1 be extended to functions w^0, w^1 defined for *all* $j\in\mathbf{Z}$ according to

$$w_{-j}^\sigma=(-1)^j w_j^\sigma \qquad \sigma=0,1,\ \ j\in\mathbf{Z}. \tag{6.4}$$

Let $\{v^n\}$, $0\leqq n<\infty$, be the solution obtained under $\bar{Q}$ with initial data v^0, v^1, and let $\{w^n\}$, $0\leqq n<\infty$, be the solution obtained under Q with initial data w^0, w^1. We claim that, for all $n\geqq 0$, v^n is identical to the restriction of w^n to $j\geqq -1$:

$$w_j^n=v_j^n \quad \text{for} \quad j\geqq -1,\ \ n\geqq 0. \tag{6.5}$$

To establish this, it is enough to show that w^n, like v^n, satisfies the boundary condition (6.3) for all n, i.e., $w_{-1}^n=-w_1^n$. But this is a corollary of the more general fact, which is easily proved by induction, that w^n satisfies (6.4) for all $\sigma\geqq 0$.

Now since leap frog with $\lambda<1$ is Cauchy-stable, one has from (3.7)

$$\|w^n\|\leqq M(\|w^0\|+\|w^1\|) \tag{6.6}$$

for some M (dependent on λ). Together with (6.4) this implies

$$\|v^n\|_+\leqq\|w^n\|\leqq M(\|w^0\|+\|w^1\|)\leqq\sqrt{2}M(\|v^0\|_++\|v^1\|_+), \qquad n\geqq 0,$$

which establishes (6.1).

Remark. The above proof has the unfortunate feature that (6.3) is not consistent with any mathematically reasonable boundary condition for the equation $u_t=u_x$. Example β of Section 2 was more satisfactory in this respect. We suspect it too is l^2-stable, but have not managed to show this.

The next result, in conjunction with Theorem 3a, proves that the distinction between finite and infinite reflection coefficients also has a real effect on l^2-instability.

THEOREM 2b. *There exists a model $\bar{Q}$ admitting a strictly rightgoing steady-state solution for which*

$$\|S^n\| \leqq \text{const.}\sqrt{n} \quad \textit{for all} \quad n \geqq 1. \tag{6.7}$$

Proof: This proof is a variation on the last one. Consider again the domain $j \geqq -1$, and take $\bar{Q}$ to be leap frog (6.2) together with the boundary condition

$$v_{-1}^{n+1} = v_1^{n+1}. \tag{6.8}$$

The reflection coefficient function (cf. (2.6)) for this problem is

$$\frac{a_r}{a_l} = -\frac{\kappa_l^2 - 1}{\kappa_r^2 - 1} = \frac{\kappa_l^2 - 1}{1 - 1/\kappa_l^2} = \kappa_l^2\left(\frac{\kappa_l^2 - 1}{\kappa_l^2 - 1}\right),$$

which has modulus 1 whenever $|\kappa_l| = 1$, but $\bar{Q}$ admits two sawtoothed strictly rightgoing steady-state solutions, $v_j^n = (-1)^j$ and $v_j^n = (-1)^n$, both of which have $C = +1$.

Given initial data v^0, v^1 satisfying (6.8), let $N \geqq 2$ be chosen. Let v^0 and v^1 be extended to functions w^0 and w^1 defined for all $j \in \mathbf{Z}$ by the conditions

$$w_j^\sigma = \begin{cases} 2v_0^\sigma - v_{|j|}^\sigma & \text{for } -N \leqq j \leqq -2,\ j \text{ even}, \\ v_{|j|}^\sigma & \text{otherwise}. \end{cases} \qquad \sigma = 0, 1, \tag{6.9}$$

Let w^n for $n \geqq 2$ be the solution obtained under (6.2) with initial data w^0, w^1. We claim that, for $n \leqq N$, v^n is identical to the restriction of w^n to $j \geqq -1$:

$$w_j^n = v_j^n \quad \text{for} \quad j \geqq -1, \qquad 0 \leqq n \leqq N. \tag{6.10}$$

To establish this, it is enough to show that w^n, like v^n, satisfies the boundary condition

$$w_{-1}^n = w_1^n, \qquad 0 \leqq n \leqq N. \tag{6.11}$$

We have constructed w in (6.9) to make this so. In fact, (6.11) is a special case of the symmetry property

$$w_{-j}^n = w_0^n + (-1)^j(w_0^n - w_j^n), \qquad 0 \leqq n \leqq N, \quad |j| \leqq \min\{N, N+1-n\}.$$

This identity holds for $n = 0, 1$ by (6.9) and for subsequent n by the following induction argument based on the formula (6.2):

$$\begin{aligned} w_{-j}^{n+1} &= w_{-j}^{n-1} + \lambda(w_{-j+1}^n - w_{-j-1}^n) \\ &= w_0^{n-1} + (-1)^j(w_0^{n-1} - w_j^{n-1}) - (-1)^j\lambda(-w_{j-1}^n + w_{j+1}^n) \\ &= w_0^{n-1} + (-1)^j(w_0^{n-1} - w_j^{n+1}) \\ &= w_0^{n+1} + (-1)^j(w_0^{n+1} - w_j^{n+1}), \qquad n \leqq N-1. \end{aligned}$$

In the last step we have used the fact that $w_0^{n+1} = w_0^{n-1}$, which follows from (6.11) applied at the previous time step together with (6.2).

Now by the triangle inequality one has from (6.9)

$$\|w^\sigma\| \leqq \sqrt{2}\|v^\sigma\|_+ + 2(\tfrac{1}{2}Nh)^{1/2}|v_0^\sigma| \leqq \sqrt{2}(1+\sqrt{N})\|v^\sigma\|_+ \leqq 3\sqrt{N}\|v^\sigma\|_+ .$$

Combining this with (6.6) and (6.10) yields the growth rate bound

$$\|v^N\|_+ \leqq \|w^N\| \leqq M(\|w^0\| + \|w^1\|) \leqq 3M\sqrt{N}(\|v^0\|_+ + \|v^1\|_+),$$

and since N was arbitrary, this implies (6.7).

A similar trick provides a growth rate bound for a problem with an infinite reflection coefficient:

THEOREM 3b. *There exists a model $\bar{Q}$ admitting a strictly rightgoing steady-state solution with infinite reflection coefficient for which*

$$\|S^n\| \leqq \text{const.}\, n \quad \textit{for all} \quad n \geqq 1. \tag{6.12}$$

Proof: Consider (6.2) on $j, n \geqq 0$ with boundary condition

$$v_0^{n+1} = v_1^{n+1}. \tag{6.13}$$

The reflection function (cf. (2.6)) is now

$$\frac{a_r}{a_l} = \frac{\kappa_l - 1}{1 - \kappa_r} = \frac{\kappa_l - 1}{1 + 1/\kappa_l}$$

which becomes infinite for $\kappa_l = -\kappa_r = -1$, corresponding to the strictly rightgoing steady-state solution $v_j^n = (-1)^n$ with $C = +1$.

Let extensions w^0, w^1 of v^0, v^1 be defined as follows:

$$w_j^\sigma = \begin{cases} v_j^\sigma, & j \geqq 0, \\ 0, & j \leqq -N, \\ 2\sum_{i=1}^{|j|} (-1)^{i+1} w_i^\sigma + (-1)^j w_{-j+1}^\sigma, & -N+1 \leqq j \leqq -1, \end{cases} \qquad \sigma = 0, 1. \tag{6.14}$$

As usual, let w^n be computed for $n \geqq 2$ by applying (6.2) for all $j \in \mathbf{Z}$. Let (6.14) be rewritten in the equivalent form

$$w_j^n - w_{j-1}^n = (-1)^{j+1}[w_{-j+1}^n - w_{-j+2}^n], \qquad -N+2 \leqq j \leqq -1, \; n = 0, 1. \tag{6.15}$$

It is readily verified from (6.13) and (6.14) that (6.15) holds (with $n = 0, 1$) not just for $j \leqq -1$, but for all j in the range $-N+2 \leqq j \leqq N$. We claim further that, in fact, (6.15) holds for all j, n satisfying $0 \leqq n \leqq N, |j-1| \leqq \min\{N-1, N-n\}$.

The proof is again an induction based on (6.2):

$$\begin{aligned} w_j^{n+1} - w_{j-1}^{n+1} &= w_j^{n-1} - w_{j-1}^{n-1} + \lambda(w_{j+1}^n - w_{j-1}^n - w_j^n + w_{j-2}^n) \\ &= (-1)^{j+1}(w_{-j+1}^{n-1} - w_{-j+2}^{n-1}) + \lambda(-1)^{j+1}(-w_{-j}^n + w_{-j+1}^n + w_{-j+2}^n - w_{-j+3}^n) \\ &= (-1)^{j+1}(w_{-j+1}^{n+1} - w_{-j+2}^{n+1}). \end{aligned}$$

In particular, (6.15) applies with $j=1$ for $0 \leqq n \leqq N$, where it reduces to the boundary condition of $\bar{Q}$:

$$w_0^n = w_1^n \qquad 0 \leqq n \leqq N.$$

Therefore the restriction of w to $j \geqq 0$ equals v for $n \leqq N$,

$$(6.16) \qquad w_j^n = v_j^n, \qquad j \geqq 0, \quad 0 \leqq n \leqq N.$$

Now for each j in the range $-N+1 \leqq j \leqq -1$, we have, by (6.14) and the discrete Hölder inequality,

$$|w_j^\sigma| \leqq 2 \sum_{i=1}^N |v_i^\sigma| \leqq 2\left(N \sum_{i=1}^N |v_i^\sigma|^2\right)^{1/2}, \qquad j \leqq -1, \quad \sigma = 0, 1,$$

which implies

$$h \sum_{j=-N+1}^{-1} |w_j^\sigma|^2 \leqq 4N(N-1)\|v^\sigma\|_+^2, \qquad \sigma = 0, 1,$$

and therefore

$$\|\omega^\sigma\| \leqq (4N(N-1)+1)^{1/2}\|v^\sigma\|_+ \leqq 2N\|v^\sigma\|_+ .$$

Combining this with (6.6) and (6.16) gives

$$\|v^N\|_+ \leqq \|w^N\| \leqq M(\|w^0\| + \|w^1\|) \leqq 2MN(\|v^0\|_+ + \|v^1\|_+),$$

which implies (6.12).

We turn now to sharpness theorems for stability with respect to boundary data. We have already seen in Lemma 4.2 that, for any model $\bar{Q}$, the operators $S_{bc}^{(n)}$ can grow no faster than $\sqrt{n}S^n$, so our first two results, which assert nothing stronger than this, are consequences of the corresponding theorems already established for l^2-stability.

THEOREM 4b. *There exists a model $\bar{Q}$ admitting a rightgoing steady-state solution (i.e., GKS-unstable) for which*

$$(6.17) \qquad \|S_{bc}^{(n)}\| \leqq \text{const.}\sqrt{n} \quad \textit{for all} \quad n \geqq 1.$$

Proof: The proof of Theorem 1b exhibited a model $\bar{Q}$ with $\|S^n\| \leqq \text{const.}$ Applying Lemma 4.2 shows that (6.17) must hold for the same $\bar{Q}$.

THEOREM 5b. *There exists a model $\bar{Q}$ admitting a strictly rightgoing steady-state solution for which*

$$\|S_{bc}^{(n)}\| \leqq \text{const. } n \quad \textit{for all} \quad n \geqq 1.$$

Proof: Apply Lemma 4.2 to the example of Theorem 2b, which satisfies $\|S^n\| \leqq \text{const.} \sqrt{n}$.

Our final theorem (which includes Theorem 5b as a restricted case) is not trivial like the last two, since the growth rate in question is no longer larger by a factor $\sqrt{n}$ than that for the initial data problem (Table I).

THEOREM 6b. *There exists a model $\bar{Q}$ admitting a strictly rightgoing steady-state solution with infinite reflection coefficient for which*

$$\|S_{bc}^{(n)}\| \leqq \text{const. } n \quad \textit{for all} \quad n \geqq 1. \tag{6.18}$$

Proof: We repeat the trick of the proof of Theorem 3b, combining it with the discrete Duhamel principle. Consider for $\bar{Q}$ the leap frog formula (6.2) with homogeneous initial data $f^0 = f^1 = 0$ and the boundary condition

$$v_0^{n+1} = v_1^{n+1} + g^n, \qquad n \geqq 1. \tag{6.19}$$

From the proof of Theorem 3b, we know that $\bar{Q}$ has an infinite reflection coefficient at $\kappa_l = -\kappa_r = -1$. Given g and some $N \geqq 2$, let us construct distributions $f^2, \cdots, f^N$ for all $j \in \mathbf{Z}$ according to the formulas $f^2 = 0$ and

$$f_j^{n+1} = \begin{cases} -\lambda g^{n-1}, & j = 0, 1, \\ -2\lambda g^{n-1}, & -N+3 \leqq j \leqq -1, \\ 0 & \text{otherwise}. \end{cases} \qquad 2 \leqq n \leqq N-1, \tag{6.20}$$

Now each f^n satisfies $f_0^n = f_1^n$, and moreover it is easy to verify that if f^n is inserted as initial data at step n for (6.2) applied on $(-\infty, \infty)$, this equality will persist for $N-3$ time steps, hence through time step N. (Each f^n satisfies (6.15) for $|j-1| \leqq N-3$.) Adding all of these solutions together, consider the distribution $\{w^n\}$ defined by $w^0 = w^1 = 0$ and

$$w_j^{n+1} = w_j^{n-1} + \lambda(w_{j+1}^n - w_{j-1}^n) + f_j^{n+1}, \qquad n \geqq 1.$$

By construction we have

$$v_j^n = w_j^n + \delta_{0j} g^{n-1} \qquad 2 \leqq n \leqq N, \quad j \geqq 0 \tag{6.21}$$

because the sum on the right, like v^n, satisfies (6.2) for $j \geqq 1$ (the case $j = 1$ must be checked separately) and (6.19) for $j = 0$.

On the other hand, we compute

$$\|f^{n+1}\|_2^2 = h\lambda^2(4N-10)|g^{n-1}|^2,$$

and, by applying the discrete Hölder inequality to the Cauchy stability bound (6.6),

$$\|w^N\|^2 \leqq M^2(\|f^3\| + \cdots + \|f^N\|)^2 \leqq M^2(N-2)(\|f^3\|^2 + \cdots + \|f^N\|^2).$$

Combining these two results and applying (4.8) yields

$$\|w^N\|^2 \leqq M^2(N-2)h\lambda^2(4N-10)\sum_{\sigma=1}^{N-2}|g^\sigma|^2 \leqq 4M^2(N-1)^2\lambda\|g\|_t^2;$$

hence

$$\|w^N\| \leqq 2MN\sqrt{\lambda}\|g\|_t.$$

Together with (6.21) this implies

$$\|v^N\| \leqq (2MN\sqrt{\lambda} + 1)\|g\|_t,$$

which establishes (6.18).

Acknowledgments. Jonathan Goodman has provided much valuable advice in the course of this work, and also contributed to the proofs of Section 5. Marsha Berger, William Coughran, Gerald Hedstrom, Robert Higdon, Peter Lax, Randall LeVeque, and Joseph Oliger also deserve thanks for their assistance.

This work was supported by a Visiting Fellowship at the Center for Mathematical Analysis, Australian National University, June 1982, by a National Science Foundation Postdoctoral Fellowship, and by the U.S. Department of Energy under Contract DE-ACO2-76-ERO3077-V.

Bibliography

[1] Brenner, P., Thomée, V., and Wahlbin, L. B., *Besov Spaces and Applications to Difference Methods for Initial Value Problems*, Springer Lect. Notes in Math. 434, 1975.

[2] Brillouin, L., *Wave Propagation and Group Velocity*, Academic, New York, 1960.

[3] Coughran, W. M., Jr., *On the Approximate Solution of Hyperbolic Initial-Boundary Value Problems*, PhD diss., Dept. of Computer Sci., Stanford U., 1980.

[4] Gustafsson, B., *The convergence rate for difference approximations to mixed initial boundary value problems*, Math. Comp. 29, 1975, pp. 396–406.

[5] Gustafsson, B., Kreiss, H.-O. , and Sundström, A., *Stability theory of difference approximations for mixed initial boundary value problems. II*, Math. Comp. 26, 1972, pp. 649–686.

[6] Harten, A., Hyman, J., and Lax, P., *On finite-difference approximations and entropy conditions for shocks*, Comm. Pure Appl. Math. 29, 1976, pp. 297–322.

[7] Higdon, R., *Initial-boundary value problems for linear hyperbolic systems*, to appear.

[8] Iserles, A., and Strang, G., *The optimal accuracy of difference schemes*, Trans. Amer. Math. Soc., to appear.

[9] Kreiss, H.-O., *Difference approximations for the initial-boundary value problem for hyperbolic differential equations*, in Numerical Solution of Nonlinear Diff. Eqs., D. Greenspan, (Ed.), Proc. Adv. Symp. Math. Res. Ctr., U of Wisconsin, Wiley, 1966.

[10] Kreiss, H.-O., *Stability theory for difference approximations of mixed initial boundary value problems. I*, Math. Comp. 22, 1968, pp. 703–714.

[11] Kreiss, H.-O., *Initial boundary value problems for hyperbolic systems*, Comm. Pure Appl. Math. 23, 1970, pp. 277–298.
[12] Kreiss, H.-O., *Boundary conditions for hyperbolic differential equations*, Proc. Dundee Conf. on Numer. Soln. of Diff. Eqs., Springer Lect. Notes in Math. 363, 1974.
[13] Kreiss, H.-O., and Oliger, J., *Methods for the Approximate Solution of Time Dependent Problems*, Global Atmospheric Research Programme Publ. Ser. 10, Geneva, 1973.
[14] Lax, P., and Nirenberg, L., *On stability for difference schemes; a sharp form of Gårding's inequality*, Comm. Pure Appl. Math. 19, 1966, pp. 473–492.
[15] Lighthill, J., *Waves in Fluids*, Cambridge University Press, Cambridge, 1978.
[16] Michelson, D., *Stability theory of difference approximations for multidimensional initial-boundary value problems*, Math. Comp. 40, 1983, pp. 1–45.
[17] Ortega, J., *Numerical Analysis: A Second Course*, Academic, New York, 1972.
[18] Osher, S., *Stability of difference approximations of dissipative type for mixed initial-boundary value problems, I*, Math. Comp. 23, 1969, pp. 335–340.
[19] Osher, S., *Systems of difference equations with general homogeneous boundary conditions*, Trans. Amer. Math. Soc. 137, 1969, pp. 177–201.
[20] Richtmyer, R. D., and Morton, K. W., *Difference Methods for Initial-Value Problems*, 2nd ed., Wiley-Interscience, New York, 1967.
[21] Sakamoto, R., *Mixed problems for hyperbolic equations*, J. Math. Kyoto U. 10, 1970, pp. 349–373 and pp. 403–417.
[22] Thomée, V., *Stability theory for partial difference operators*, SIAM Review 11, 1969, pp. 152–195.
[23] Trefethen, L. N., *Group velocity in finite difference schemes*, SIAM Review 24, 1982, pp. 113–136.
[24] Trefethen, L. N., *Wave Propagation and Stability for Finite Difference Schemes*, PhD diss., Dept. of Computer Sci., Stanford U., 1982.
[25] Trefethen, L. N., *Group velocity interpretation of the stability theory of Gustafsson, Kreiss, and Sundström*, J. Comp. Phys. 49, 1983, pp. 199–217.
[26] Trefethen, L. N., *On L^p-instability and dispersion at discontinuities under finite difference formulas*, to appear.
[27] Trefethen, L. N., *Stability of hyperbolic finite difference models with one or two boundaries*, Proc. AMS–SIAM 1983 Summer Seminar on Large-Scale Computations in Fluid Mechanics, S. Osher (ed.), to appear.
[28] Vichnevetsky, R., and Bowles, J., *Fourier Analysis of Numerical Approximations of Hyperbolic Equations*, SIAM, Philadelphia, 1982.
[29] Whitham, G., *Linear and Nonlinear Waves*, Wiley-Interscience, New York, 1974.
[30] Gustafsson, B., report in Proc. AMS–SIAM 1983 Summer Seminar on Large-Scale Computations in Fluid Mechanics, S. Osher (ed.), to appear.

Received December 1982.