Notes of a Numerical Analyst

Random Fibonacci sequences

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Random Fibonacci sequences are generated by the recurrence $x_{n+1} = x_n \pm x_{n-1}$, where each $\pm$ is an independent coin toss. Let's look at the "semi-Fibonacci" variant

$$x_{n+1} = x_n \pm \tfrac{1}{2} x_{n-1}. \qquad (1)$$

If you run (1) for a few steps, you may get a perplexing result, as in Figure 1.

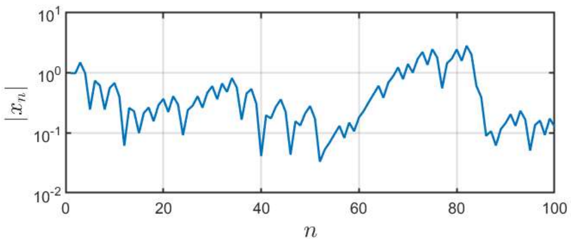

Figure 1. 100 steps of (1) with $x_1 = x_2 = 1$.

With more steps, however, as in Figure 2, the pattern becomes clear. *Random semi-Fibonacci sequences decrease exponentially.*

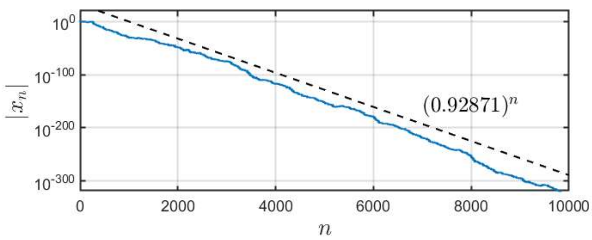

Figure 2. 10000 steps.

This effect is so elemental that it can illustrate many ideas of mathematics and science. (Warning! — all details below omitted.)

Geometric means. Some intuition for the decrease goes like this. Suppose $\{x_n\}$ were of approximately constant magnitude. Then each step would multiply the magnitude at random by $\frac{1}{2}$ or $\frac{3}{2}$. But the geometric mean of $\frac{1}{2}$ and $\frac{3}{2}$ is less than 1, suggesting decay on average after all.

Almost sure behaviour of random processes. In principle, $\{x_n\}$ could decay or grow at any rate from $(0.707)^n$ to $(1.366)^n$, but with probability 1, you'll see $(0.929)^n$ as $n \to \infty$. It is effects like this that give meaning to quantities of physics starting with pressure, temperature, and entropy.

Lyapunov constants. The Lyapunov constant of this dynamical system is 0.92871. One could hardly devise a simpler illustration of this idea.

Fractals. Analysis of (1) reveals an invariant measure of fractal form, and if $\frac{1}{2}$ is generalized to a parameter β, the dependence of the Lyapunov constant on β is also fractal.

Products of random matrices $[1 \pm\frac{1}{2}; 1\ 0]$. This topic was made famous by Furstenberg and Kesten.

Stochastic differential equations. (1) is related to approximations to SDEs with multiplicative noise like the exponential martingale $dX_t = \sigma X\, dW_t$. Here, too, solutions decay exponentially.

Heavy-tailed distributions. Though x_n decreases exponentially, its variance and standard deviation grow exponentially! At the endpoint of Figure 2, the standard deviation is 10^{677}.

Some numbers can't be determined analytically. I believe the digits 0.92871 are correct, but it would be astonishing if anyone found an exact formula.

FURTHER READING

[1] M. Embree and L. N. Trefethen, Growth and decay of random Fibonacci sequences, *Proc. Roy. Soc. Lond. A* 455 (1999), 2471–2485.

[2] D. Viswanath, Random Fibonacci sequences and the number $1.13198824\ldots$, *Math. Comput.* 69 (2000), 1131–55.

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