Why structured eigenvalue perturbation analysis may be inappropriate for analyzing robustness

Nick Trefethen, Oxford University
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Many systems are structured, and if they are perturbed, the perturbations are structured too. It seems obvious that therefore, in studying perturbations of eigenvalues to assess robustness of a system, we should ideally look at structured perturbations. Nevertheless I think this conclusion is unjustified.

Consider first the problem of *distance to singularity* (see p. 447 of *Spectra and Pseudospectra*). If A is a nonsingular matrix, will a random perturbation make it singular? No, the probability of this is zero! So if the distance to singularity $\varepsilon(A)$ of a matrix A is interesting, this cannot be because a perturbation might make it singular. What's going on then? I think that $\varepsilon(A)$ is interesting because it is a proxy for something else: it tells us how sensitive solutions of the problem $Ax = b$ are to perturbations in the data, since $\varepsilon(A) = 1/\|A^{-1}\|$. The connection between distance to singularity and condition numbers was made famous by Kahan and Demmel.

Since $\|A^{-1}\|$ is invariant under orthogonal change of coordinates, it is structure-independent. It follows that the unstructured distance to singularity is the one mainly of use.

Now consider *distance to instability*, that is, eigenvalues in the right half-plane. If A is stable, might a random perturbation make it unstable? Ah, yes it might, with a positive probability if the perturbation is big enough! This fact, I suspect, is a red herring—like the unstable eigenvalue at $\mathrm{Re} = 5772$ in plane Poiseuille flow, which has nothing to do with the instabilities that actually appear in these flows. I suspect that what we really mainly want to know about in practice is how sensitive solutions of $\dot{u} = Au + b(t)$ are to perturbations in the data. Again, it would seem that this question is structure-independent, and so should be any means by which we answer it.

There's a special case of this where structured analysis is certainly not enough: *real vs. complex perturbations* for $\dot{u} = Au + b(t)$. If A and b and u are real, shouldn't you ideally consider real perturbations of this system rather than complex? No! It is the perturbations of the eigenvalues of A under complex perturbations that tell you something about the behavior of A, even as applied to real inputs. See the 2×2 example on pp. 455–457 of *Spectra and Pseudospectra.*

As problems get more complicated (higher-order systems, distance to controllability, hamiltonian structure, etc.) I am further and further from my expertise. Maybe the analogy with distance to singularity is not a good guide here, and one truly needs structured perturbation analysis? Maybe; maybe not. I don't think it is enough to say that the relevance of structured perturbations is "obvious".

Here is my best guess. I think looking at eigenvalues of perturbed systems is a powerful and convenient procedure that tells us something about robustness. For a complicated and perhaps even nonlinear system, it may easier to look at perturbations than to figure out a truly "right" robustness analysis. But ultimately the former works because it's a proxy for the latter.