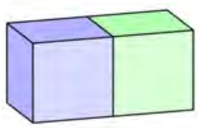
Notes of a Numerical Analyst

# Two Cubes

NICK TREFETHEN FRS

Here's a mathematical problem. You have two unit cubes face to face against each other, each a uniform solid of mass 1. According to Newton's inverse-square gravitational law with gravitational constant 1, what's the force of attraction $F$ between them?



I doubt you've encountered this problem, and it's easy to see why. It looks pointless. There are no serious applications, and no deeper mathematical ideas are lurking here. The force is given by a six-dimensional integral with respect to the coordinates $x, y, z$ of one cube and $\xi, \eta, \zeta$ of the other:

$$F = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_1^2 f \; dx\, dy\, dz\, d\xi\, d\eta\, d\zeta$$

with

$$f = \frac{x - \xi}{\left[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right]^{3/2}}.$$

The integral is intractable, and it seems there's not much to say about it.

I cooked up this problem some years ago as a challenge for the graduate students in Oxford's Numerical Analysis "Problem Solving Squad". Could they calculate $F$ to 10 digits of accuracy? Six-dimensional integrals are easy enough if the integrand is smooth — you can use six-fold Gauss quadrature, for example. But this integrand is singular since the cubes are touching, and calculating $F$ accurately is hard. The elegant trick we found, devised by Alex Prideaux, was to break each cube into eight sub-cubes, whereupon a step of recursion eliminates the singularity.

The answer is $F \approx 0.9259812605\ldots$, and I know just one interesting thing about this number: it is less than 1. If the cubes were balls or spheres, $F$ would be exactly 1, as Newton proved. Here we get $F < 1$ because cubes have some of their mass further out in the corners.

But then came a surprise from Bengt Fornberg of the University of Colorado, who is a wizard of formulae as well as algorithms. Fornberg spent several days gnawing on the integral. It's not hard to reduce the dimensionality from 6 to 3, but he managed to reduce it further to dimension 2, and then 1. Finally, he astonished us with an exact result:

$$\begin{aligned} F = \frac{1}{3}\Big(&\frac{26\pi}{3} - 14 + 2\sqrt{2} - 4\sqrt{3} + 10\sqrt{5} - 2\sqrt{6} \\ &+ 26\log(2) - \log(25) + 10\log(1+\sqrt{2}) \\ &+ 20\log(1+\sqrt{3}) - 35\log(1+\sqrt{5}) \\ &+ 6\log(1+\sqrt{6}) - 2\log(4+\sqrt{6}) \\ &- 22\tan^{-1}(2\sqrt{6})\Big). \end{aligned}$$

Look at this mess! There are 14 terms, each one as arbitrary as $10\log(1+\sqrt{2})$. How could we even be confident it was correct? Of course, ironically, by checking it against the numerical approximation. Yes, this freight train of a formula really is the right answer.

Does it matter if a problem has an exact solution? If there were no explicit formula, would that make a difference to the status of the two cubes problem? Or if the answer were simply $1/\sqrt{2}$?

## FURTHER READING


[1] F. Bornemann et al., The SIAM 100-Digit Challenge: A Study in High-Accuracy Numerical Computing, SIAM, 2004.

[2] W. Hackbusch, Direct integration of the Newton potential over cubes, Computing, 68 (2002), 193–216.

[3] L. N. Trefethen, Ten digit problems, in D. Schleicher and M. Lackmann, eds., Invitation to Mathematics: From Competitions to Research, Springer, 2011.



**Nick Trefethen**

Trefethen is Professor of Numerical Analysis and head of the Numerical Analysis Group at the University of Oxford.