Dirac - von Neumann axioms in the setting of Continuous Model Theory

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Abstract

We recast the well-known axiom system of quantum mechanics used by physicists (the Dirac calculus) in the language of Continuous Logic. For the basic version of the axiomatic system we prove that along with the canonical continuous model the axioms have approximate finite models of large sizes, in fact the continuous model is isomorphic to an ultraproduct of finite models. We analyse the continuous logic quantifier corresponding to Dirac integration and show that in finite context it has two versions, local and global, which coincide on Gaussian wave-functions.

1 Introduction

1.1 The axiomatic formulation of quantum mechanics was introduced by Paul Dirac in 1930 [1] through a description of Hilbert space, and later developed with greater mathematical rigor in a monograph of 1932 by John von Neumann. Since 1930, Dirac went through several rewritings and new editions to refine his calculus to a level he considered satisfactory. In fact, von Neumann himself expressed his dissatisfaction not long after publication of his book, and spent considerable effort looking for alternatives [20]. Modern books present Dirac's axioms in a succinct form, often omitting much of the technical detail.

In section 2 we survey the axioms of quantum mechanics following [2]. Readers with a background in logic will notice that what physicists refer to as axioms is very far from what is a conventional set of axioms in a formal language even in its early form as presented e.g. by Hilbert's axiomatisation of geometry [3]. We noted earlier, in [5] and in [7], that the language that Dirac introduced is that of *continuous logic*, CL, and the (rigged) Hilbert space which the axioms describe is a close analogue of cylindric algebra of Tarski, see [6], in a CL-version. In section 4 we go further and prove our main Theorem 4.7 stating that the continuous theory of Dirac - von Neumann given in Hilbert-space form (with integral operators) has two kinds of models: the canonical model $\mathbb{U}(\infty)$ based on wave-functions/continuous predicates on $\mathbb{R}^m$, and an asymptotic class of finite approximate models which we call H-structures, $\mathbb{U}(n)$. More precisely we prove that the canonical continuous model $\mathbb{U}(\infty)$ is isomorphic to a CL-ultraproduct of the finite $\mathbb{U}(n)$, $n \in \mathbb{N}$.

As a matter of fact our $\mathbb{U}(n)$ are specific lattice models. Of course, a variety of lattice models have been in use in physics. They play an important role as toy models, as vehicles for specific calculations as well as modelling specific phenomena. However, the fact that the specific cyclic lattice models $\mathbb{U}(n)$ hosting the full set of integral operators represent in mathematically exact way Dirac's axioms of quantum mechanics is novel. In particular, the Dirac integral is shown to correspond to a summation formula which is "local" in a certain well-defined sense, namely the summation domain has to be much shorter then the length of the full cycle. All unitary operators of the form e^{iL} where L is self-adjoint in a form of a polynomial of P and Q with rational coefficients, are represented in the $\mathbb{U}(n)$.

The proof of the theorem is based on an analysis of Dirac's integral (equivalently of the structure of the underlying rigged Hilbert space) in the context of a CL-quantifier. Modelling Dirac's integral in finite CL-structures $\mathbb{U}(n)$ we expose a specific *local* nature of this quantifier and compare it with another possible *global* quantifier, based on discrete number-theoretic Gauss summation. Our second main result Theorem 5.6 states that for theories restricted to Gaussian states (whose Hamiltonian includes quadratic terms only) the local and the global quantifiers act equivalently.

Gaussian part of quantum mechanics is the backbone of the theory. From it the theory extends by considering perturbed Gaussian states. We demonstrate in subsection 5.7 that the global quantifier is applicable to perturbed Gaussian states with result close to ones of perturbation theory.

2 Dirac's calculus and axiomatisation of quantum mechanics

Below we reproduce a slighly edited version of axioms from [2], 6.3.

2.1 Axiom 1. The state of a quantum system is described by a vector $|\psi\rangle$ belonging to a complex Hilbert space $\mathcal{H}$. This state is usually called "ket ψ". A complex Hilbert space $\mathcal{H}$ is a vector space, which can be finite dimensional or infinite dimensional, equipped with the complex scalar product (also called inner product) $\langle\psi|\psi'\rangle$ between any pair of states $|\psi\rangle$, $|\psi'\rangle$ in $\mathcal{H}$. The norm, or modulus, of a generic vector $|\psi\rangle \in \mathcal{H}$ is defined as

$$||\psi|| = |\langle\psi|\psi\rangle|$$

and usually $|\psi\rangle$ is normalized to one, i.e.$||\psi|| = 1$. The symbol $\langle\psi|$ which appears in the definition of the norm is called "bra ψ" and it can be intepreted as the fuction

$$\langle\psi| : \mathcal{H} \to \mathbb{C}.$$

For any $|\psi'\rangle \in \mathcal{H}$ this function gives a complex number $\langle\psi|\psi'\rangle$ obtained as scalar product of $|\psi\rangle$ and $|\psi'\rangle$. In a complex Hilbert space $\mathcal{H}$ it exists a set of basis vectors $|\phi_\alpha\rangle$ which are orthonormal, i.e. $\langle\phi_\alpha|\phi_\beta\rangle = \delta(\alpha - \beta)$, and such that

$$|\psi\rangle = \sum_\alpha c_\alpha |\phi_\alpha\rangle \tag{1}$$

for any $|\psi\rangle$, where the coefficients c_α belong to $\mathbb{C}$.

Axiom 2. Any observable (measurable quantity) of a quantum system is described by a self-adjoint linear operator $F : \mathcal{H} \to \mathcal{H}$ acting on the Hilbert space of state vectors.

For any classical observable F it exists a corresponding quantum observable F.

Axioms 3.The possible measurable values of an observable F are its eigenvalues f, such that

$$F|f\rangle = f|f\rangle$$

with $|f\rangle$ the corresponding eigenstate. The observable $|f\rangle$ admits the spectral resolution

$$F = \sum_f f|f\rangle\langle f| \tag{2}$$

where $\{|f\rangle\}$ is the set of orthonormal eigenstates of F, and the mathematical object $\langle f|$, called "bra of f", is a linear map that maps into the complex number. This also satisfy the identity

$$\sum_f |f\rangle\langle f| = \mathrm{I}.$$

Axiom 4. The probability P of finding the state $|\psi\rangle$ in the state $|f\rangle$ (both of norm 1) is given by

$$P = |\langle f|\psi\rangle|^2$$

This probability P is also the probability of measuring the value f of the observable F when the system is in the quantum state $|\psi\rangle$.

Axiom 5. The time evolution of states and observables of a quantum system with Hamiltonian H is determined by the unitary operator

$$K^t := \exp(-i\mathrm{H}t/\hbar)$$

, such that $|\psi(t)\rangle = K^t|\psi\rangle$ is the time-evolved state $|\psi\rangle$.

2.2 The dynamical reformulation of quantum mechanics. This is based on the Stone Theorem:

For each self-adjoint operator A on $\mathcal{H}$ there is a well-defined one parameter group of unitary operators on $\mathcal{H}$

$$\{\mathrm{e}^{iAt} : \ t \in \mathbb{R}\}$$

and A can be recovered uniquely from the group.

Thus, we may reduce the theory to the equivalent theory of Hilbert spaces with unitary operators of the form above. One advantage of such a theory is that the unitary operators, unlike unbounded self-adjoint operators, are defined on the whole of $\mathcal{H}$ and their treatment is mathematically more straightforward. The framework is also called the Heisenberg picture of quantum mechanics.

2.3 Now we make several comments on the axioms.

The term "Hilbert space" here should actually be read as *the rigged Hilbert space* (see [11]) because it differs from the standard definition by accommodating both a Hilbert space Φ and the dual space Φ^* with

$$\Phi \subseteq \mathcal{H} \subseteq \Phi^*.$$

The summation formulas like (1) and (2) are presented in a form of an integral if the family $|\psi_\alpha\rangle$ is continuous but seems natural in the summation form when α runs in the discrete spectrum of an operator.

2.4 Remark. Rigged Hilbert spaces provide a powerful mathematical framework to extend quantum mechanics, allowing distributions and generalized eigenfunctions to be rigorously handled. However, as is almost generally accepted, not every element corresponds to a physically realisable state – some are purely mathematical artifacts, see e.g. [12].

In the more general context of quantum field theories Wightman axioms explicitly postulate that physically meaningful part of the rigged Hilbert space $\mathcal{H}$ is a dense subset $\mathcal{D} \subset \mathcal{H}$.

3 The axioms in the setting of Continuous Logic

3.1 We discuss in this section the possible interpretation of the above axioms in terms of (the most general versions of) Continuous Logic (CL) and Continuous Model Theory.

Recall that in a most general terms the language of CL consists of *predicate* symbols (we will ignore function symbols for now), a collection of *connectives*, that is continuous functions $\mathbb{C}^n \to \mathbb{C}$, and quantifiers, that is continuous transformations of predicates.

A basic CL-formula is made of predicate symbols using connectives and quantifiers.

An interpretation of symbols and formulas begins with a choice of a *universe* M, which may be a metric space or, in more recent application, a measure space.

Symbols of n-ary predicates P are interpreted as maps $P : M^n \to \mathbb{C}$. If $f : \mathbb{C}^n \to \mathbb{C}$ is a connective and $\psi_1, \ldots, \psi_n$ are formulas, equivalently, definable predicates, than the formula $f(\psi_1, \ldots, \psi_n)$ is interpeted as the composition of the maps defined by $\psi_1, \ldots, \psi_n$ with $f(x_1, \ldots, x_n)$. Quantifiers are interpreted in a special way as transformations of formulas in $n+1$ variables into formulas in n variables.

The uniformity of interpretation of language symbols across different M is ensured by certain uniform continuity moduli for the symbols P.

A universe M together with interpretation of predicates P of the language constitutes a *structure* in continuous model theory. Importantly, the definable sets in a structure are obtained not just by CL-formulas but also as limits in the families of formulae-definable sets.

See [4], [14] and [**?**] for further details.

3.2 Recall that the historical prototype of a vector $|\psi\rangle$ of the Hilbert space has been a wave-function, that is a function

$$\psi : \mathcal{M} \to \mathbb{C}$$

from a configuration space $\mathcal{M}$ into a bounded domain of the complex numbers $\mathbb{C}$.

These can be seen as predicates on a domain which, as the matter of facr is identified in Dirac calculus of quantum mechanics with $\mathbb{R}^n$, where $\mathbb{R}$ is the real line seen as a measure space. Definable predicates of norm 1 will be referred to as states.

Of special significance are the **momentum and position states**. Momentum states where defined by Dirac as the definable family of predicates of the form

$$|p\rangle := \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-ipx}, \quad p \in \mathbb{R}. \tag{3}$$

One can consider a $\mathbb{C}$-linear space generated by the momentum states and define Hermitian inner product, first between the momentum states

$$\langle p_1 | p_2 \rangle := \delta(p_1 - p_2) \tag{4}$$

where δ is the Dirac delta. However, in the context of rigged Hilbert spaces one can identify the inner product above with the Kronecker delta.

The **position** states $|x\rangle$, $x \in \mathbb{R}$, by their physical meaning are characteristic functions of one-point subsets $\{x\}$

$$|x\rangle := \delta(x - z) \tag{5}$$

(as a function of z) which for convenience of continuous mathematical manipulations have been replaced here and in (4) by Dirac's delta-functions, that is by distributions. In this sense

$$|x\rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{ixp} |p\rangle dp \tag{6}$$

Equivalently, position states can be represented by linear functionals (bra-vectors)

$$\langle x| : \ |\psi\rangle \mapsto \psi(x)$$

or equivalently, for ψ running in $\{|p\rangle \ : \ p \in \mathbb{R}\}$,

$$\langle x| : \ |p\rangle \mapsto \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-ipx} \tag{7}$$

In model theory terms, the linear functionals $\langle x|$ are imaginary elements in the structure, the interpretation of which is given by (7).

The basic unitary operators (Weyl operators) can be defined by their action on the basis:

$$\mathrm{e}^{i\mathrm{P}} : |p\rangle \mapsto \mathrm{e}^{ip}|p\rangle$$
$$\mathrm{e}^{i\mathrm{Q}} : |x\rangle \mapsto \mathrm{e}^{ix}|x\rangle.$$

In particular, the former can be equivalently, using (6), written as

$$\mathrm{e}^{i\mathrm{P}} : |x\rangle \mapsto \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{ip(x+1)} |p\rangle dp$$

The time-evolution operator K^t_{Free} for a free particle is $\mathrm{e}^{-it\frac{\mathrm{P}^2}{2}}, \ t \in \mathbb{R}$, that is

$$K^t : |p\rangle \mapsto \mathrm{e}^{-it\frac{p^2}{2}} |p\rangle \tag{8}$$

which yields by (6)

$$K^t : |x_0\rangle \mapsto \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{i(px_0 - t\frac{p^2}{2})} |p\rangle dp$$

and

$$\langle x|K^t|x_0\rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{i(px_0 - t\frac{p^2}{2})} \langle x|p\rangle dp$$

Substituting (3) one gets

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{i(px_0 - t\frac{p^2}{2})} \langle x|p\rangle dp = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{i(p(x_0 - x) - t\frac{p^2}{2})} dp =$$

$$\frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-it\frac{(p - \frac{(x-x_0)}{t})^2}{2} + \frac{i(x-x_0)^2}{2t}} dp = \frac{1}{2\pi} \mathrm{e}^{i\frac{(x-x_0)^2}{2t}} \cdot \left(\int_{\mathbb{R}} \mathrm{e}^{-it\frac{p^2}{2}} dp \right)$$

$$= \frac{1}{2\pi} \mathrm{e}^{i\frac{(x-x_0)^2}{2t}} \cdot \sqrt{\frac{2\pi}{it}} = \frac{1}{\sqrt{2\pi i t}} \mathrm{e}^{\frac{i(x-x_0^2)}{2t}}$$

and one obains the well-known formula

$$\langle x|K^t|x_0\rangle = \frac{1}{\sqrt{2\pi i t}} \mathrm{e}^{\frac{i(x-x_0^2)}{2t}} \tag{9}$$

3.3 In a more abstract axiomatic setting the theory of Dirac integration in the context of Gaussian states can be expressed in formulae:

for $a \neq 0$,

$$\int_{\mathbb{R}} \mathrm{e}^{\pi i(ax^2+2bx)} dx = \sqrt{\frac{1}{ia}} \mathrm{e}^{-\pi i \frac{b^2}{a}} = \mathrm{e}^{-\frac{\pi i}{4}} \sqrt{\frac{1}{a}} \mathrm{e}^{-\pi i \frac{b^2}{a}} \tag{10}$$

and,

$$\int_{\mathbb{R}} \mathrm{e}^{-2\pi i b x} dx = \delta(b) = b^{-1}\delta(0) \tag{11}$$

More advanced calculus, going beyond Gaussian states, requires methods of perturbation theory. This can be illustrated by the following typical calculation related to *anharmonic oscillator* in quantum mechanics and also setting a pattern for crucial calculations in QFT

$$\int_{\mathbb{R}} \mathrm{e}^{i\frac{x^2+\lambda x^4}{h}} dx = \mathrm{e}^{\frac{\pi i}{4}} \sqrt{2\pi h}(1 + i\lambda h + o(\lambda h)) \tag{12}$$

for $h > 0$ small, see [17], section 2.

3.4 In terms of structures, let $\mathcal{H}_m$ be the set of all m-ary predicates on $\mathbb{R}$. This by definition has structure of $\mathbb{C}$-vector spaces

$$\mathbb{C} = \mathcal{H}_0 \subset \ldots \subset \mathcal{H}_m \subset \ldots \subset \mathcal{H}_{m+1} \ldots \mathcal{H}.$$

Also, one uses quantifiers, linear maps written as integrals

$$\phi(z_1, \ldots, z_n) \mapsto \int_{\mathbb{R}} \phi(z_1, \ldots, z_n) dz_n.$$

In fact, this is a collection of linear maps

$$\int : \mathcal{H}_{m+1} \to \mathcal{H}_m,$$

the rules of calculation of which defined by Dirac's improper integration.

For a discrete basis the sum in (1) can be represented in CL-setting as a definable countable sum, see e.g. [14], Example 5.2.

A special binary operation in the spaces, inner product,

$$\mathcal{H}_m \times \mathcal{H}_m \to \mathbb{C}; \quad \langle \phi(z_1, \ldots, z_n), \psi(z_1, \ldots, z_n) \rangle = \int_{\mathbb{R}^m} \phi^* \cdot \psi \, dz_1 \ldots dz_m$$

where ϕ^* is the complex comjugate of ϕ and $\int_{\mathbb{R}^m}$ is m-multiple integral. $\langle \phi | \psi \rangle$ can be seen as a continuous predicate of equality $\phi = \psi$.

One restricts the notion of **state** to predicates ϕ such that $\langle \phi | \phi \rangle = 1$.

An important role in the theory is played by a collection of linear maps (operators)

$$L : \mathcal{H}_m \to \mathcal{H}_m$$

with physical meanings. These can be of the integral form

$$\phi(\bar{z}_1, \bar{z}_2) \mapsto \int_{\mathbb{R}^k} \alpha(\bar{y}, \bar{z}_1) \cdot \phi(\bar{y}, \bar{z}_2) \; d\bar{y}$$

where $|\bar{y}| = |\bar{z}_1| = k$, $\alpha \in \mathcal{H}_{2k}$, or as classical linear operators

$$\mathrm{P} : \phi(x, \bar{z}) \mapsto i\hbar \frac{\partial \phi(x, \bar{z})}{\partial x} \quad \text{or} \quad \mathrm{Q} : \phi(x, \bar{z}) \mapsto x \cdot \phi(x, \bar{z})$$

The **time evolution operator** $\exp(-i\mathrm{H}t/\hbar)$ acts on states as a unitary operator determining the evolution of a state in time t with a given Hamiltonian H. A state ϕ_{t_0} determining a system at time t_0 evolves into a state $\phi_t := \exp(-i\mathrm{H}(t - t_0)/\hbar)$ with the *probability amplitude* equal to $\langle \phi_{t_0} | \phi_t \rangle$, which is a complex number of modulus 1. The calculation of the CL-formulae ϕ_t and $\langle \phi_{t_0} | \phi_t \rangle$ (which involve mainly calculations of the application of quantifier $\int$) is the central problem of quantum theory, equivalent to solving the associated Schrödinger equation.

All of the above together makes the $\mathcal{H}_m$ a collection of Hilbert spaces with linear operators and $\mathcal{H}$ an ambient Hilbert space.

Note that unitary operators of the form e^{iL}, where L is self-adjoint can mostly be represented in the integral form with respective kernels α. In particular, it is true when L is a polynomial of the basic operators P and Q, see [19].

It should be mentioned that the Hilbert state formalism of quantum mechanics can be fully reduced to the unitary setting, that is the setting with a Hilbert space equipped with unitary operators only. This is our preferred formalism and by the remark above, with enough functions α in the formalism, one can reduce all the operators to the integration operator.

Below we explain how the Hilbert space axiomatisation of QM can be represented as a formal theory in the language of Continuous Logic.

3.5 Remarks on Dirac integration and measure. Let $\int_R f(x)\delta_x$ stand for the Dirac integral and $\int_R f(x)dx$ for the proper Riemann integral.

$$\int_R f(x)\delta_x = \int_R f(x)dx \tag{13}$$

if the latter is well-defined.

In particular, for $f(x)$ continuous on $\mathbb{R}$,

$$\int_R f(x)\delta_x = \lim_{m\to\infty}\int_{-m}^{m} f(x)dx \tag{14}$$

if the right-hand-side is well-defined.

If a finite limit in (14) does not exists the integral is understood in the sense of distributions, in particular one writes

$$\int_{\mathbb{R}} \mathrm{e}^{-2\pi bx}\delta_x = \delta(b)$$

However, in the setting of rigged Hilbert spaces it is consistent to renormalise to the Kronecker delta-symbol:

$$\int_{\mathbb{R}} \mathrm{e}^{-2\pi bx}\delta_x := \delta^{\mathrm{Kr}}_{0,b}. \tag{15}$$

3.6 Remarks on rigged Hilbert spaces. .

Recall (see e.g. [9]) that a Gelfand triple is:

$$\Phi \subset \mathcal{H} \subset \Phi^*$$

where Φ is a space of test functions (e.g. the space of continuous functions on $\mathbb{R}$ with compact support), $\mathcal{H}$ is the Hilbert space, Φ^* is the space of continuous linear functionals on Φ, i.e., distributions.

In the context of continuous model theory with universe $\mathbb{R}$ it is natural to take for Φ continuous functions which are zero outside the interval $[-m, m]$, which agrees with (14) and further remarks above.

An element $\phi \in \Phi^*$ acts on a test function f via the application of inner product in Φ^*: $f \mapsto \langle f|\phi\rangle$

The ket $|x\rangle \in \Phi^*$ is not a vector in $\mathcal{H}$, but a generalized eigenvector.

The pairing $\langle x|\phi\rangle$ can be interpreted as the evaluation $\phi(x)$ of ϕ at the point x, if such evaluation makes sense. If $\phi \in \mathcal{H}$, this is a well-defined function $\phi : \mathbb{R} \to \mathbb{C}$

4 Hilbert space formalism and H-structures

4.1 The axiomatic description of quantum mechanical theory in the form of rigged Hilbert space may be quite confusing from the logician point of view – there are no logical sentences which can be called axioms. What Axioms 1 – 5 render instead is the topological-algebraic structure of a Hilbert space with operators. This brings us to the *algebraisation of logic* approach introduced by A.Lindenbaum, A.Tarski, P.Halmos for the first order setting. It is quite natural to see the Hilbert space formalism as the form of algebraic logic in the context of the continuous logic of physics.

As explained in our less formal note [5] the Hilbert space $\mathcal{H}$ of quantum mechanics, or rather the tower of Hilbert spaces $\mathcal{H}^{\otimes n}$, plays the role of the Tarski *cylindric algebra*, see [6].

4.2 Let $\mathcal{H}$ be a rigged Hilbert space, $\mathcal{H}_n = \mathcal{H}^{\otimes n}$ and O a collection of linear operators on the $\mathcal{H}_n$. Let $\mathcal{H}_n^{\mathrm{Def}}$ is a dense subspace of $\mathcal{H}_n$ closed under operators from O.

We will often write $\mathcal{H}$ for the union of the tower $\mathcal{H}_1 \subset \mathcal{H}_2 \subset \ldots$ and similarly with $\mathcal{H}^{\mathrm{Def}}$.

Definition. An H-**structure** $(\mathbb{U}, \mathcal{H}^{\mathrm{Def}}, \mathrm{O})$ is given by

- a **universe** $\mathbb{U}$, a complete metric space with a measure μ and metric $\mathrm{dist}(u_1, u_2)$.

In case $\mathbb{U} = \mathbb{U}(\mathfrak{n})$ is finite of size $\mathfrak{n}$, we identify

$$\mathbb{U}(\mathfrak{n}) := [-\frac{\mathfrak{n}}{2}, \frac{\mathfrak{n}}{2}) \cap \mathbb{Z}$$

with additive structure isomorphic to $\mathbb{Z}/\mathfrak{n}\mathbb{Z}$ and $\mathrm{dist}(u_1, u_2) = \sqrt{\frac{1}{\mathfrak{n}}} \cdot |u_1 - u_2|$, the measure of a point is $\frac{1}{\sqrt{\mathfrak{n}}}$.

In case $\mathbb{U} = \mathbb{U}(\infty)$ is infinite, $\mathbb{U}(\infty) := \mathbb{R}$, $\mathrm{dist}(u_1, u_2) = |u_1 - u_2|$ and the measure is the Dirac's delta-measure as determined in 3.5.

- collections H_n, $n \in \mathbb{N}$, of **predicates**

$$\psi : \mathbb{U}^n \to \mathbb{C}$$

each with a name ψ from $\mathcal{H}_n^{\mathrm{Def}}$, continuous maps; H_n has a structure of a $\mathbb{C}$-linear space;

- an **Hermitian inner product** $\langle \psi_1 | \psi_2 \rangle$; $H_n \times H_n \to \mathbb{C}$ is defined for all n;

- **quantifier**

$$\mathrm{E} : H_{n+1} \to H_n,$$

for all $n \in \mathbb{N}$, which is given as an integral operator

$$\mathrm{E} : \psi \mapsto \int_{\mathbb{U}} \psi \, d\mu$$

on the rigged Hilbert space H_{n+1}.

- a collection of **linear operators**

$$L : H_n \to H_m$$

named by symbols $L \in \mathrm{O}$:

- **Weyl assumption:** O contains a pair of (Weyl) operators U and V, acting on H_1 and satisfying the commutation relation

$$\mathrm{UV} = q\mathrm{VU}$$

where

$$q = \begin{cases} \mathrm{e}^{\frac{2\pi i}{\mathcal{N}}} \text{ if } \mathbb{U} \text{ is finite } |\mathbb{U}| = \mathcal{N} \\ \mathrm{e}^{2\pi i \mathfrak{h}} \text{ some } \mathfrak{h} \in \mathbb{R}_{>0} \end{cases}$$

There is a canonical set of eigenvectors for U

$$\mathrm{Eig}\,\mathrm{U} = \{\mathbf{u}[r] \in H_1 : \mathrm{U}\mathbf{u}[r] = q^r \cdot \mathbf{u}[r],\ r \in \mathbb{U}\} \quad \langle \mathbf{u}[r] | \mathbf{u}[s] \rangle = \delta_{r,s}$$

which form an orthonormal **basis** of H_1. The action of V on the basis is defined as

$$\mathrm{V}\mathbf{u}[r] = \mathbf{u}[r+1],\ \text{where } \mathbf{u}[r] = \mathbf{u}[s],\ \text{if } |\mathbb{U}| = \mathfrak{n} \text{ and } r \equiv s \bmod \mathcal{N}.$$

There is a dual canonical set of eigenvectors for V with eigenvalues in $\mathbb{S}(\mathbb{U}) \subset \mathbb{S}$

$$\mathrm{Eig}\,\mathrm{V} = \{\mathbf{v}[p] \in H_1 : \mathrm{V}\mathbf{v}[p] = q^p \cdot \mathbf{v}[p],\ p \in \mathbb{U}\} \quad \langle \mathbf{v}[p] | \mathbf{u}[q] \rangle = \delta_{p,q}$$

We call an infinite H-structure a **continuous** H-structure. Recall that $\mathbb{U} = \mathbb{U}(\infty) = \mathbb{R}$ in this case.

4.3 There is a Fourier duality between the bases $\mathrm{Eig}\,\mathrm{U}$ and $\mathrm{Eig}\,\mathrm{V}$, in finite cases:

$$\mathbf{v}[p] = \frac{1}{\sqrt{\mathfrak{n}}} \sum_{r \in \mathbb{U}} q^{rp} \mathbf{u}[r]; \quad \mathbf{u}[r] = \frac{1}{\sqrt{\mathfrak{n}}} \sum_{p \in \mathbb{U}} q^{-rp} \mathbf{v}[p] \tag{16}$$

or, in continuous/rigged Hilbert spaces form, in agreement with (5),

$$\mathbf{v}[p] = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-irp}; \quad \mathbf{u}[r] = \delta(r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathbf{v}[p] \mathrm{e}^{ipr} dp$$

In the setting of continuous model theory we represent the one-point characteritic functions $\mathbf{u}[r]$ as the limit of continuous bump-functions.

4.4 We also consider asymptotic classes of H-structures with given signature $\mathcal{H}^{\mathrm{Def}}, \mathrm{O}$, which are classes of H-structures with finite universe $\mathbb{U}(n)$, n running in a subset $P \subseteq \mathbb{N}$. (Similar to the class considered by Hrushovski in [15]). Then, given an ultrafilter $\mathcal{D}$ on P and the first-order ultraproduct

$$\mathbb{U}(\mathfrak{n}) = \prod_{n \in \mathcal{D}} \mathbb{U}(n)$$

we obtain a pseudo-finite version of H-structure. We study these below along with CL-ultraproducts.

We say that $(\mathbb{U}, \mathcal{H}^{\mathrm{Def}}, \mathrm{O})$ **represents** the (possibly uncomplete) Hilbert space $(\mathcal{H}^{\mathrm{Def}}, \mathrm{O})$ if

$$(H, \mathrm{O}) \cong (\mathcal{H}^{\mathrm{Def}}, \mathrm{O})$$

via the naming correspondence.

4.5 Proposition. *Given a rigged Hilbert space $\mathcal{H}$ with an inner product subspace $\mathcal{H}^{\mathrm{Def}}$ which is dense in $\mathcal{H}$ and closed under the quantifier and a* *family of unitary operators* O *satisfying the Weyl assumtion, there is a unique H-structure* $(\mathbb{U}, \mathcal{H}^{\mathrm{Def}}, \mathrm{O})$ *representing* $(\mathcal{H}^{\mathrm{Def}}, \mathrm{O})$ *.*

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Proof. Set $\mathbb{U} := \mathrm{Eig}\,\mathrm{U}$ and define, for each $\psi \in \mathcal{H}_n^{\mathrm{Def}}$, an n-ary predicate ψ to be the unique map $\psi : \mathbb{U}^n \to \mathbb{C}$ such that

$$\psi(\bar{r}) = \langle \mathbf{u}[\bar{r}] \,|\, \psi \rangle$$

This is well-defined according to 3.6 (note that $\mathbf{u}[\bar{r}] = |\bar{r}\rangle$, a position state in $\mathcal{H}_n$). $\square$

4.6 CL-ultraproduct. We will work in a specific asymptotic class of H-structures of signature $(\mathcal{H}^{\mathrm{Def}}, \mathrm{E})$ where the predicates of $\mathcal{H}^{\mathrm{Def}}$ on $\mathbb{U}(\mathfrak{n})$ are of the form

$$\psi(k, \ldots, k_m) = f(\frac{k_1}{\sqrt{\mathfrak{n}}}, \ldots, \frac{k_m}{\sqrt{\mathfrak{n}}}),$$

where $f(x_1, \ldots, x_m)$ is a smooth function $\mathbb{R}^m \to \mathbb{C}$.

The predicate with the same name ψ on continuous $\mathbb{U}$ is

$$\psi(x_1, \ldots, x_m) := f(x_1, \ldots, x_m).$$

The quantifier $\mathrm{E} := \{\mathrm{E}^{(m)} : m \in \mathbb{N}\}$, a family of quantifiers on finite intervals of diameter $2m < \sqrt{\mathfrak{n}}$, is defined as follows:

$$\mathrm{E}_k^{(m)} \psi(k, \bar{p}) = \mathrm{st}\left(\frac{1}{\sqrt{\mathfrak{n}}} \sum_{k \geq -m\sqrt{\mathfrak{n}}}^{k \leq m\sqrt{\mathfrak{n}}} \psi(k, \bar{p}) \right), \text{ for } \mathbb{U}(\mathfrak{n}) \tag{17}$$

$$\mathrm{E}_x^{(m)} \psi(x, \bar{y}) = \int_{-m}^{m} \psi(x, \bar{y}) dx, \text{ for } \mathbb{U}(\infty). \tag{18}$$

We also consider:

$$\mathrm{E}_k^{\mathrm{loc}} \psi(k, \bar{p}) := \lim_{m \to \infty} \mathrm{E}_k^{(m)} \psi(k, \bar{p}) \tag{19}$$

Here, in case $\mathfrak{n}$ is finite, the limit should be understood as the value for the maximal m satisfying $2m \leq \sqrt{\mathfrak{n}}$.

For a pseudo-finite $\mathfrak{n}$ it is assumed that m runs in $\mathbb{N}$, the standard positive integers. In the context of continuous model theory $\mathrm{E}_k^{\mathrm{loc}}$ is definable in terms of $\mathrm{E}_k^{(m)}$.

While $\mathrm{E}_k^{(m)}\psi$ is well-defined for all ψ, $\mathrm{E}_k^{\mathrm{loc}}\psi$ might be not, for some ψ for infinite $\mathfrak{n}$.

However, we use notation $\mathrm{E}^{\mathrm{loc}}$ for the family $\{\mathrm{E}^{(m)} : m \in \mathbb{N}\}$ when it does not lead to confusion.

4.7 Theorem. $(\mathcal{H}, \mathrm{O})$ *be a rigged Hilbert space with a family* O *of integral operators, and* $\mathcal{H}^{\mathrm{Def}}$ *its dense subspace closed under inner product and* O.

For any non-principal ultrafilter $\mathcal{D}$ *on* $\mathbb{N}$ *the continuous model theory ultraproduct* $(\mathbb{U}^*, \mathcal{H}^{\mathrm{Def}}, \mathrm{O})$ *of finite H-structures* $(\mathbb{U}(n), \mathcal{H}^{\mathrm{Def}}, \mathrm{O})$ *is a continuous H-structure.*

For every sentence σ *and every positive* ϵ *there is a subset* $D_{\sigma,\epsilon} \in \mathcal{D}$ *such that for all* $n \in D_{\sigma,\epsilon}$ *the value of* σ *on* $(\mathbb{U}, \mathcal{H}^{\mathrm{Def}}, \mathrm{O})$ *differs from the value of* σ *on* $(\mathbb{U}(n), \mathcal{H}^{\mathrm{Def}}, \mathrm{O})$ *by no more than* ϵ.

Proof. The metric universe $\mathbb{U}^*$ of the ultraproduct is defined as the union of sorts of finite diameter $2m$, which are limits along the ultafilter $\mathcal{D}$ of sorts of the same diameter of $\mathbb{U}(n)$. This means that for a limit non-standard number $\mathfrak{n}$ and numbers $k \in \mathbb{U}(\mathfrak{n})$ we set the limit point $x = k/_{\mathcal{D}}$ so that

$$\mathrm{dist}(0, x) = \mathrm{dist}(0, k)/_{\mathcal{D}}$$

This brings us to

$$x := \mathrm{st}(\frac{k}{\sqrt{\mathfrak{n}}}),$$

(the standard part map). In particular, the interval $[-m\sqrt{\mathfrak{n}}, m\sqrt{\mathfrak{n}}]$ in $\mathbb{U}(\mathfrak{n})$ corresponds to the interval $[-m, m]$ in $\mathbb{R}$.

This also agrees with the definition of predicates ψ on the ultraproduct and

Next we prove the correspondence for quantifiers. It is enough to consider unary $\psi : \mathbb{R} \to \mathbb{C}$. By definition $\psi(k) = f(\frac{k}{\sqrt{\mathfrak{n}}})$, $f(x)$ smooth on $\mathbb{R}$.

Claim. Given any positive $\epsilon \in \mathbb{R}$,

$$|\frac{1}{\sqrt{\mathfrak{n}}} \sum_{-m\sqrt{\mathfrak{n}} \le k < m\sqrt{\mathfrak{n}}} f(\frac{k}{\sqrt{\mathfrak{n}}}) \; - \; \int_{-m}^{m} f(x)dx| < \epsilon$$

Indeed, the discrete formula is a Riemann sum with spacing $\Delta x = \frac{1}{\sqrt{\mathfrak{n}}}$. By the left Riemann sums estimate for an interval (a, b)

$$\mathrm{Err} \le M_f \frac{(a-b)^2}{2N} = M_f \frac{(2m)^2}{4m\sqrt{\mathfrak{n}}}$$

where N is the number of points between $a = m$ and $b = -m$ and $M_f = \max\{f'(x) : b \leq x < a\}$. Clearly, $M_f \frac{(2m)^2}{4m} \in \mathbb{R}$ and thus $M_f \frac{(2m)^2}{4m\sqrt{\mathfrak{n}}} < \epsilon$ because $\frac{1}{\sqrt{\mathfrak{n}}}$ is a non-standard infinitesimal.

Thus the application of the quantifier in the asymptotic class agrees with the quantifier in the ultraproduct. It follows that the inner product operation in the asymptotic class agrees with the inner product operation in the ultraproduct, once it is determined by integration. This is enough to obtain the correspondence for the construction of interpretable linear functionals and the rigged Hilbert space in the asymptotic class and in the ultraproduct. It follows that the inner product operation $\langle\psi_1|\psi_2\rangle$ is preserved by the ultraproduct for all $\psi_1, \psi_2 \in \mathcal{H}^{\mathrm{Def}}$.

Finally, the operators in O are preserved by the ultraproduct because they are expressible in terms of $\mathrm{E}^{\mathrm{loc}}$. This includes Weyl operators.

□

5 Gaussian and perturbation-Gaussian states

5.1 Gaussian predicates.

Call an m-predicate $\psi(k_1, \ldots, k_m)$ on $\mathbb{U}(\mathfrak{n})$ basic Gaussian if there is a $\eta \in \mathbb{C}$ and a positive-definite quadratic form $Q(x_1, \ldots, x_m)$ over $\mathbb{Q}$ such that

$$\psi(k_1, \ldots, k_m) = \eta \cdot \mathrm{e}^{-\pi i \frac{Q(k_1, \ldots, k_m)}{\mathfrak{n}}} = \eta \cdot \mathrm{e}^{-\pi i Q(\frac{k_1}{\sqrt{\mathfrak{n}}}, \ldots, \frac{k_m}{\sqrt{\mathfrak{n}}})}$$

For the continuous $\mathbb{U}$:

$$\psi(x_1, \ldots, x_m) = \eta \cdot \mathrm{e}^{-\pi i Q(x_1, \ldots, x_m)},$$

where Q is a over $\mathbb{R}$.

Since by definition e^{-irp} is a Gaussian predicate, we consider the Fourier-dual one point characteristic function $\mathbf{u}[r]$ to be a Gaussian state.

Note that $Q(x_1, \ldots, x_m)$ can be written in the form that singles out a particular variable, say x_1,

$$Q(x_1, \ldots, x_m) = ax^2 + 2xb(\bar{y}) + c(\bar{y})$$

where $x = x_1$, $\bar{y}$ is the rest of the variables, $b(\bar{y})$ a linear form and $c(\bar{y})$ a quadratic form.

Now a Gaussian predicate can be written as

$$\psi(k,\bar{p}) = \eta \cdot \mathrm{e}^{-\pi i \frac{c(\bar{p})}{\mathfrak{n}}} \cdot \mathrm{e}^{-\pi i \frac{ak^2+2kb(\bar{p})}{\mathfrak{n}}}$$

in the discrete setting, and

$$\psi(x,\bar{y}) = \eta \cdot \mathrm{e}^{-\pi i c(\bar{y}))} \cdot \mathrm{e}^{-\pi i (ax^2+2xb(\bar{y})}$$

in continuous seting.

For the discrete version, if $a \neq 0$, we call the rational number a the **period of** ψ with respect to variable k.

If $a = 0$ and $b(\bar{p}) = b \cdot (L_1 p_1 + \ldots + L_m p_m)$ with $L_1, \ldots, L_m$ coprime tuple of integers, then the **period of** ψ with respect to variable k is equal to b.

Definitions. Call non-standard integer $\mathfrak{n}$ **highly divisible** if it is divisible by all standard integers.

Let $\mathfrak{n}$ be highly divisble and $|\mathbb{U}| = \mathfrak{n}$. We say that a subset $X \subset \mathbb{U}^m$ is **d-dense** if X contains a submodule of $\mathbb{U}^m$ of finite index.

5.2 Lemma (Gauss summation). *On $\mathbb{U}(\mathfrak{n})$, for $\mathfrak{n}$ highly divisible:*

$$\frac{1}{\sqrt{\mathfrak{n}}} \sum_{-\frac{\mathfrak{n}}{2a} \leq k < \frac{\mathfrak{n}}{2a}} \mathrm{e}^{-\pi i \frac{ak^2+2kb(\bar{p})}{\mathfrak{n}}} = \sqrt{\frac{1}{a}} \cdot \mathrm{e}^{\frac{\pi i}{4}} \cdot \mathrm{e}^{-\pi i \frac{b(\bar{p})^2}{a\mathfrak{n}}} \tag{20}$$

for all $\bar{p}$ in a d-dense subset of $\mathbb{U}^{m-1}$, and equals 0 outside the d-dense subset.

In case $a = 0$ and $b(\bar{p}) = b \cdot p$, for $b \in \mathbb{Q}$,

$$\frac{1}{\sqrt{\mathfrak{n}}} \sum_{-\frac{\mathfrak{n}}{2b} \leq k < \frac{\mathfrak{n}}{2b}} \mathrm{e}^{\pi i \frac{2bkp}{\mathfrak{n}}} = b^{-1} \delta^{(\mathfrak{n})}(p) \tag{21}$$

where

$$\delta^{(\mathfrak{n})}(p) = \begin{cases} 0 & \text{if } p \neq 0 \\ \sqrt{\mathfrak{n}} & \text{otherwise} \end{cases}$$

Proof. Let $a = \frac{A}{D} > 0$ where $A, D \in \mathbb{Z}$. We choose D so that $D \cdot b(\bar{p})$ is over $\mathbb{Z}$. Note that $\frac{\mathfrak{n}}{2a}$ is an integer because $\mathfrak{n}$ is divisible by A by assumptions.

Now let

$$X_a = \{\bar{p} \in \mathbb{U}^{m-1} : A | D b(\bar{p})\}.$$

This is a dense subset of $\mathbb{U}^{m-1}$. For a $\bar{p} \in X_a$ the function

$$\mathrm{e}^{-\pi i \frac{ak^2+2kb(\bar{p})}{\mathfrak{n}}}$$

of variable k has period $\frac{\mathfrak{n}}{a}$ and $\frac{b(\bar{p})}{a}$ is an integer. Thus the summands in

$$ak^2 + 2kb(\bar{p}) = a(k + \frac{b(\bar{p})}{a})^2 - \frac{b(\bar{p})^2}{a} = an^2 - \frac{b(\bar{p})^2}{a}$$

are integer and we can write

$$\sum_{-\frac{\mathfrak{n}}{2a} \le k < \frac{\mathfrak{n}}{2a}} \mathrm{e}^{-\pi i \frac{ak^2+2kb(\bar{p})}{\mathfrak{n}}} = \mathrm{e}^{i\pi \frac{\frac{b(\bar{p})^2}{a}}{\mathfrak{n}}} \sum_{0 \le n < \frac{\mathfrak{n}}{a}} \mathrm{e}^{-\pi i \frac{an^2}{\mathfrak{n}}} = \mathrm{e}^{i\pi \frac{b(\bar{p})^2}{a\mathfrak{n}}} \cdot \sqrt{\frac{\mathfrak{n}}{a}} \mathrm{e}^{\frac{\pi i}{4}}$$

where at the last step we used the classical Gauss' quadratic sums equality.

In case $\bar{p} \notin X_a$ the Gauss sum is equal 0.

Now consider the case $a = 0$, $b = \frac{B}{D}$, for $B, D \in \mathbb{N}$ coprime. If $p = 0 \mod B$ then all the summands in (21) are equal to 1 and we get $\frac{\sqrt{\mathfrak{n}}}{b}$ for the value of the formula. Alternatively, if $p \neq 0 \mod B$ then we get all the roots of 1 of order $\frac{\mathfrak{n}}{b}$ as summands, and the sum is equal 0.

□

5.3 Lemma. *For any d-dense subset $X \subset \mathbb{U}(\mathfrak{n})^m$ for any $\bar{y} \in \mathbb{R}^m$ there is $\bar{p} \in X$ such that* $\mathrm{st}(\frac{1}{\sqrt{\mathfrak{n}}}\bar{p}) = \bar{y}$.

Proof. When p runs in $\mathbb{U}(\mathfrak{n}) = [-\frac{\mathfrak{n}}{2}, \frac{\mathfrak{n}}{2}]$ the numbers $\mathrm{st}(\frac{1}{\sqrt{\mathfrak{n}}}p)$ run continuously between $-\infty$ and $+\infty$. Thus there is $\bar{p} \in \mathbb{U}(\mathfrak{n})^m$ such that $\mathrm{st}(\frac{1}{\sqrt{\mathfrak{n}}}\bar{p})$.

Density of X implies that there is a tuple of non-negative standard integers $\bar{d}$ such that $\bar{p} + \bar{d} \in X$. But $\mathrm{st}(\frac{1}{\sqrt{\mathfrak{n}}}\bar{d}) = \bar{0}$ and thus we can assume $\bar{p} \in X$.

□

5.4 Corollary

Let $(\mathbb{U}^*, \mathcal{H}^{\mathrm{Def}}, \mathrm{O})$ *be the ultraproduct constructed in 4.7,* $\mathfrak{n}$ *highly divisible, and* $\mathrm{e}^{-\pi i a x^2 + 2xb(\bar{y})}$ *a Gaussian predicate on* $\mathbb{U}^*$*, where* a, b *are rational. Then,*

$$\frac{1}{\sqrt{\mathfrak{n}}} \sum_{-\frac{\mathfrak{n}}{2a} < k \le \frac{\mathfrak{n}}{2a}} \mathrm{e}^{-\pi i \frac{ak^2+2kb}{\mathfrak{n}}} = \int_{\mathbb{R}} \mathrm{e}^{-\pi i (ax^2+2xb)} dx \qquad (22)$$

if $a > 0$, *and*

$$\frac{1}{\sqrt{\mathfrak{n}}}\sum_{-\frac{\mathfrak{n}}{2b}<k\leq\frac{\mathfrak{n}}{2b}} \mathrm{e}^{-\frac{2\pi ikb}{\mathfrak{n}}} - \int_{\mathbb{R}} \mathrm{e}^{-2\pi ixb}dx \tag{23}$$

where we dropped $\delta^{(\mathfrak{n})}$ on the left and Dirac delta on the right of equality (see (15)).

Remark. (23) makes sense when $b=0$, in which case one has a constant function of value 1 on the right and the constant sequence on the left. These are Gaussian predicates as well (the case $a=0=b$). The formula can also serve for calculating norms.

5.5 Global versions of quantifiers.

The **global** quantifier is defined for finite and pseudo-finite $\mathbb{U}$:

$$\mathrm{E}_k^{\mathrm{glob}}\psi(k) := \mathrm{st}\left(\frac{1}{\sqrt{\mathfrak{n}}}\sum_{-\frac{\mathfrak{n}}{2P}<k\leq\frac{\mathfrak{n}}{2P}}\psi(k)\right)$$

where P is the period of ψ.

5.6 Theorem. *Given a pseudo-finite H-structure with $\mathbb{U}=\mathbb{U}(\mathfrak{n})$ with $\mathfrak{n}$ highly divisible, and a Gaussian predicate ψ with variables $k,\bar{p}$:*

$$\mathrm{E}_k^{\mathrm{glob}}\psi(k,\bar{p}) = \mathrm{E}_k^{\mathrm{loc}}\psi(k,\bar{p})$$

for each $\bar{p}$ in a d-dense subset.

Proof. This is a direct consequence of 5.4 and 4.7. □

5.7 Beyond free particles. Anharmonic oscillator. The Gaussian fragment of quantum mechanics modelled above (with a little more work includes also quantum harmonic oscillator) is the only part of QM which allows exact solutions. The more general version of QM would include states of the form $\mathrm{e}^{-i\frac{x^2+f(x)}{2\mathfrak{h}})}$ where $f(x)$ is a polynomial of degree >2. In fact, the theory, due to physical and mathematical issues only deal with quite specific forms of such states. The key example is that of an anharmonic oscillator $\mathrm{e}^{-i\frac{x^2+\lambda x^4}{2\mathfrak{h}}}$ as analysed in [18]. This is also a much simplified analogue of so called ϕ^4-quantum field theory.

The important difference with the Gaussian case is that Dirac calculus over such states, namely the key calculation

$$\int_{\mathbb{R}} \mathrm{e}^{-i\frac{x^2+\lambda x^4}{2\mathfrak{h}}} dx, \quad \lambda > 0$$

can only be carried out using *perturbation* methods, which impose specific restriction on coefficients, in particular $\mathfrak{h}$ have to be infinitesimally small in the example.

This leads us to restrict our analysis to discrete states of the form

$$\psi(k) := \mathrm{e}^{-\pi i\frac{H(k^2+\frac{1}{L}k^4)}{2\mathfrak{n}}}$$

perturbed Gaussian states where H, L positive integers and

$$\mathfrak{h} = \frac{1}{2\pi H}.$$

As in the Gaussian case we assume that H divides $\mathfrak{n}$. Note that H also plays here a role of **asymptotic period** for ψ. Perturbed Gaussian states in general are not of period H.

Set $x = \sqrt{\frac{1}{\mathfrak{n}}}k$. Then[1]

$$\mathrm{e}^{-\pi i\frac{H(k^2+\frac{1}{L}k^4)}{\mathfrak{n}}} = \mathrm{e}^{-i\frac{x^2+\lambda x^4}{2\mathfrak{h}}}, \quad \lambda = \frac{\mathfrak{n}}{L}$$

5.8 As for the above Gaussian states the application of the global quantifier to a perturbed state ψ is defined as

$$\mathrm{E}_k^{\mathrm{glob}}\psi(k) := \sqrt{\frac{1}{\mathfrak{n}}} \sum_{-\frac{\mathfrak{n}}{2H} \le k < \frac{\mathfrak{n}}{2H}} \psi(k)$$

That is for ψ as above

$$\mathrm{E}_k^{\mathrm{glob}}\psi(k) := \sqrt{\frac{1}{\mathfrak{n}}} \sum_{-\frac{\mathfrak{n}}{2H} \le k < \frac{\mathfrak{n}}{2H}} \mathrm{e}^{-\pi i\frac{Hk^2+\frac{H}{L}k^4}{\mathfrak{n}}} = \sqrt{\frac{2}{\mathfrak{n}}} \sum_{0 \le k < \frac{\mathfrak{n}}{2H}} \mathrm{e}^{-\pi i\frac{Hk^2+\frac{H}{L}k^4}{\mathfrak{n}}}$$

[1]Physicists also consider perturbative states with term λx^d for $d \geq 3$. In this case $\lambda := \frac{\mathfrak{n}^{d/2-1}}{L}$.

assumin $\frac{\mathfrak{n}}{2H} > 1$ is an integer.

We will assume that
$$\lambda = O(1).$$
Then under the restriction $0 \leq k < \frac{\mathfrak{n}}{2H}$ we have
$$\frac{Hk^4}{L\mathfrak{n}} < \frac{\mathfrak{n}^3}{LH^3} = O(\frac{\mathfrak{n}^2}{H^3}) \tag{24}$$

5.9 Let $\phi(k) = 1 - \mathrm{e}(\frac{\frac{H}{L}k^4}{2\mathfrak{n}})$. Consider the partition of the sum
$$\mathrm{E}_k^{\mathrm{glob}}\psi = 2\mathfrak{h}^{\frac{1}{2}}\sqrt{\frac{H}{\mathfrak{n}}}\sum_{0\leq k<\frac{\mathfrak{n}}{2H}} \mathrm{e}^{-\pi i\frac{Hk^2+\frac{H}{L}k^4}{\mathfrak{n}}} =$$
$$= \mathfrak{h}^{\frac{1}{2}}\left(2\sqrt{\frac{H}{\mathfrak{n}}}\sum_{0\leq k<\frac{\mathfrak{n}}{2H}} \mathrm{e}^{-\pi i\frac{Hk^2}{\mathfrak{n}}} + 2\sqrt{\frac{H}{\mathfrak{n}}}\sum_{0\leq k<\frac{\mathfrak{n}}{H2}} \phi(k)\mathrm{e}^{-\pi i\frac{Hk^2}{2\mathfrak{n}}}\right) =$$
$$= \mathfrak{h}^{\frac{1}{2}}(T_0(\mathfrak{h}) + T_\phi(\mathfrak{h})).$$
We know that
$$T_0(\mathfrak{h}) = \mathrm{e}^{-\frac{\pi i}{4}}.$$
So our aim is to evaluate $T_\phi(\mathfrak{h})$.

5.10 Note that
$$\phi(k) = 1 - \mathrm{e}^{\pi i\frac{Hk^4}{L\mathfrak{n}}} = \pi i\frac{Hk^4}{L\mathfrak{n}} + O((\frac{Hk^4}{L\mathfrak{n}})^2) = \pi i\frac{\lambda Hk^4}{\mathfrak{n}^2} + \epsilon$$
Note that $k^4 \leq (\frac{\mathfrak{n}}{H})^4$ and thus $\frac{Hk^4}{\mathfrak{n}^2} \leq 2\pi\mathfrak{h}(\frac{\mathfrak{n}}{H})^2$ and $\epsilon = o(\mathfrak{h})$, so
$$|\phi(k)| \leq \mathfrak{h}\cdot O(\lambda(\frac{\mathfrak{n}}{H})^2)$$

$$T_\phi(\mathfrak{h}) = |2\sqrt{\frac{H}{\mathfrak{n}}}\sum_{0\leq k<\frac{\mathfrak{n}}{2H}} \phi(k)\mathrm{e}^{-\pi i\frac{Hk^2}{\mathfrak{n}}}| \leq |2\sqrt{\frac{H}{\mathfrak{n}}}\sum_{0\leq k<\frac{\mathfrak{n}}{2H}} \phi(k)| \leq \mathfrak{h}\sqrt{\frac{H}{\mathfrak{n}}}\cdot\frac{\mathfrak{n}}{H}\cdot O(\lambda\frac{\mathfrak{n}^2}{H^2}) =$$
$$= \lambda\mathfrak{h}\cdot O(\frac{\mathfrak{n}}{H})^{2+1/2}$$
We will say
$$\mathfrak{h} \to 0 \text{ iff } \frac{\mathfrak{n}}{H} = O(1).$$

5.11 Corollary

$$T_\phi(H) \leq O(1) \cdot \lambda\mathfrak{h} \text{ when } \mathfrak{h} \to 0.$$

5.12 Corollary

$$\mathrm{E}_k^{\mathrm{glob}}\psi(k,\mathfrak{h}) = \mathfrak{h}^{\frac{1}{2}}\sqrt{2\pi}\mathrm{e}^{\frac{\pi i}{4}}(1+I(\mathfrak{h})), \text{ where } I(\mathfrak{h}) = O(1)\cdot\lambda\mathfrak{h} \text{ when } \mathfrak{h}\to 0.$$

This is in a good agreement with $\mathrm{E}_k^{\mathrm{loc}}\psi(k,\mathfrak{h})$ calculated in [17] and [18] as an *asymptotic (non-convergent) series* of $\mathfrak{h}$.

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