

# Approximation of structures: local and global

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## 1 Introduction

**1.1** This work along with some earlier ones cited below is motivated by attempts to understand the procedures of approximation in physics, in particular by the question of how infinite and continuous emerges from finite and discrete. It is an age-old question entertained by Greek philosophers and reformulated by Hilbert in his problem 6 where he proposed developing “mathematically the limiting process ... which leads from the atomistic view to the laws of motion of continua”, see [SRaym09].

**1.2** We continue the study of structural approximation initiated in [Z14]. In [Z25-2] and [Z25-3] we realised that along with its initial *global* version there might be another useful version which we identified in a few specific cases and called *local* approximation.

Here we furnish a mathematically rigorous definition of local approximation and show how it is applicable to some interesting classes of structures. This includes the approximation of the Lorentz-invariant Minkowski space-time of [Z25-2] by finite G-invariant lattices as well as the continuous model theory approximation of quantum mechanics from [Z25-3].

Also, addressing a physics motivated question posed in [Z14], we show that local approximation of compact simple Lie groups by finite groups does exist, while [NST18] proved that a global approximation is not possible.

Our definition of local approximation reduces to the definition of the *local ultraproduct*, the ultraproduct in the class of *emerging-metric* structures. This generalises the ultraproduct of metric structures. The construction is based on the idea that in some classes of algebraic structures such as groups or fields, in sufficiently large finite structures there is a natural metric “near

the origin” but which is not necessarily well-defined globally. Under certain assumptions, in the first-order ultraproduct, the near-the-origin region will form an algebraic structure with a pseudo-metric, which then is reduced into a structure with a complete metric.

Our context includes both first order topological structures as in [Z14] and [Z25-1] and metric structures as in [BBHU08] as well as continuous logic structures as [Z25-3]. This becomes possible in the setting of *general structures* developed by Kiesler, [Kies20], which allows any predicates, not necessarily continuous, with values in the real interval  $[0, 1]$ . However, the role of the distance predicate is slightly different. In fact, our emerging metric is a generalisation of the pre-metric expansion of Kiesler.

**1.3** Applications of the new notion of approximation in the paper demonstrate that important structures of mathematical physics can be locally approximated by finite structures. In particular, in section 4 we prove that *all compact simple Lie groups can be locally approximated by finite groups*. These groups are the backbone of the Standard Model of particle physics and also of the String Theory. However, it is worth remarking that from calculational point of view the fact that a metric structure can be approximated by finite emerging metric structures probably does not add much. The main outcome of the local approximation theory is that the calculations are consistent with the assumption of finiteness of the model. Global approximation, on the other hand, may be of importance when models of closed systems like a whole universe are considered, see e.g. [Z25-2] and [Z25-3] for speculations in this direction.

In conclusion, we would like to formulate a relevant open problem.

**The Virasoro group** is the universal central extension of the infinite-dimensional Lie group of orientation preserving diffeomorphisms of the circle  $\text{Diff}^+(\mathbb{S}^1)$  by  $\mathbb{R}$ , see [Kir87]. *Is the Virasoro group or just the group  $\text{Diff}^+(\mathbb{S}^1)$  locally approximable by finite groups?*

**1.4** I am grateful to Arkady Bollobas, Simon Saunders, and Tim Palmer for fruitful discussions that helped motivate the paper.

## 2 Main definitions

Our context includes both first order topological structures as in [Z14] and [Z25-1] and metric and continuous logic structures as in [BBHU08] and

[Z25-3]. This becomes possible in the setting of *general structures* developed by Kiesler, [Kies20], which allows any predicates, not necessarily continuous, with values in the real interval  $[0, 1]$ .

**2.1 The formalism of general structures** (following [Kies20]). The space of truth values will be  $[0, 1]$ , with 0 representing truth. A vocabulary consists of a set of predicate symbols  $P$  of finite arity, a set of function symbols  $f$  of finite arity, and a set of constant symbols  $c$ . A general structure  $\mathbf{M}$  consists of a vocabulary  $V$ , a non-empty universe set  $M$ , an element  $c^M \in M$  for each constant symbol  $c$ , a mapping  $P^M : M^n \rightarrow [0, 1]$  for each predicate symbol  $P$  of arity  $n$ , and a mapping  $f^M : M^n \rightarrow M$  for each function symbol  $f$  of arity  $n$ . A general structure determines a vocabulary, but does not determine a metric signature. The formulas are as in [BBHU08], with the connectives being all continuous functions from finite powers of  $[0, 1]$  into  $[0, 1]$ , and the quantifiers  $\sup x$ ,  $\inf x$ . The truth value of a formula  $\varphi(\bar{x})$  at a tuple  $\bar{a} \in M^n$  in a general structure  $\mathbf{M}$  is an element of  $[0, 1]$  denoted by  $\varphi^{\mathbf{M}}(\bar{x})$ . It is defined in the usual way by induction on the complexity of formulas.

We are going to apply these definitions in a many-sorted version, assuming that our sorts  $S_m$ ,  $m \in \mathbb{N}$ , are nested  $S_m \subseteq S_{m+1}$ . The diameters of the sorts can grow which is not consistent with the distance predicate  $\mathbf{d} : M^2 \rightarrow [0, 1]$ . This is easily remedied by allowing a separate distance predicate  $\mathbf{d}_m$  for each sort  $S_m$  and then postulating the scaling relation between  $\mathbf{d}_m$ .

**2.2 Global approximation of first-order topological structures.** Recall the definition in [Z14] and [Z25-1].

Let  $\mathbf{M}, \mathbf{M}_{i \in I}$  be topological  $L$ -structures in a coarse topology with basic closed subsets of  $\mathbf{M}^n$ ,  $\mathbf{M}_{i \in I}^n$  defined by basic  $n$ -ary  $L$ -predicates.

Let  $\mathcal{D}$  be an ultrafilter on  $I$  and

$$*\mathbf{M} = \prod_{\mathcal{D}} \mathbf{M}_i$$

the first-order ultraproduct of  $L$ -structures. Then, following [Z14],  $\mathbf{M}$  is (structurally) approximated by the family  $\mathbf{M}_{i \in I}$  along the ultrafilter  $\mathcal{D}$  on  $I$  if there is a surjective  $L$ -homomorphism

$$\text{lm} : *\mathbf{M} \twoheadrightarrow \mathbf{M}. \tag{1}$$

In this paper we call such an approximation **global** and write (1) as

$$\text{lm}^{\text{glob}} \mathbf{M}_i = \mathbf{M} \tag{2}$$

Here and below we omit reference to the ultrafilter  $\mathcal{D}$ .

**2.3 Remark.** By [Z14], Prop.3.3, an  $\mathbf{M}$  satisfying (1) has to be quasi-compact (compact in the coarse topology), provided it is a  $T_1$ -space, that is any point of  $\mathbf{M}$  is closed. We are interested in approximating  $\mathbf{M}$  with  $T_1$ -topology and assume this property when claiming (2).

**2.4 Expansions to metric structures.** In notation of 2.2, let  $L^+$  be the vocabulary obtained from the initial  $L$  by removing the equality and adding a distance predicate  $\mathbf{d}(x_1, x_2)$ , along with unary predicates  $S_m$ ,  $m \in \mathbb{N}$ , defining new basic closed sets in  $\mathbf{M}$ .

We assume for  $\mathbf{M}$

$$\bigwedge_{m \in \mathbb{N}} S_m \subseteq S_{m+1} \quad (3)$$

and the usual properties of distance hold:

$$\mathbf{d}(x_1, x_2) \leq r_{12} \ \& \ \mathbf{d}(x_2, x_3) \leq r_{23} \rightarrow \mathbf{d}(x_1, x_3) \leq r_{12} + r_{23} \quad (4)$$

$$\bigwedge_{n \in \mathbb{N}} \mathbf{d}(x_1, x_2) \leq \frac{1}{n} \rightarrow x_1 = x_2 \quad (5)$$

Also, for each basic  $n$ -ary function  $L$ -symbol  $f(x_1 \dots x_n)$ , for each  $m$  there is a number  $k = k(m, f)$  such that

$$x_1 \dots x_n \in S_m \rightarrow f(x_1 \dots x_n) \in S_k \quad (6)$$

We assume:

$$x_1, x_2 \in S_m \rightarrow \mathbf{d}(x_1, x_2) \leq m; \quad (7)$$

for each  $\epsilon \in \mathbb{R}_{>0}$  there is  $\delta \in \mathbb{R}_{>0}$  such that

$$x_1 \dots x_n, y_1 \dots y_n \in S_m \ \& \ \bigwedge_i \mathbf{d}(x_i, y_i) < \delta \rightarrow \mathbf{d}(f(x_1 \dots x_n), f(y_1 \dots y_n)) < \epsilon \quad (8)$$

and for each basic  $n$ -ary predicate  $P$  a point  $(x_1^0 \dots x_n^0) \in S_m^n \setminus P$  there is  $\delta \in \mathbb{R}_{>0}$  such that

$$\bigwedge_i \mathbf{d}(x_i^0, x_i) < \delta \rightarrow (x_1 \dots x_n) \notin P \quad (9)$$

**2.5 Local ultraproduct and local approximation.** Let  $\mathbf{M}_{i \in I}$  be a family of  $L$ -structures that can be expanded to  $L^+$  structures satisfying (3). We say that this is a family with **emerging  $L^+$  metric** if

$$\bigcup_{m \in \mathbb{N}} S_m(M_i) = M_i : i \in I \quad (10)$$

and its first-order ultraproduct  ${}^*\mathbf{M} = \prod_{\mathcal{D}} \mathbf{M}_i$  satisfies (4), (6), (7), (8) and (9).

Define the local part of  ${}^*\mathbf{M}$

$${}^*\mathbf{M}_{\text{loc}} = \prod_{\mathcal{D}}^{\text{loc}} \mathbf{M}_i$$

to be the substructure of  ${}^*\mathbf{M}$  with the universe

$${}^*M_{\text{loc}} := {}^*M \cap \bigcup_{m \in \mathbb{N}} S_m({}^*M).$$

The definition implies that for any two points  $x_1, x_2$  in  ${}^*\mathbf{M}_{\text{loc}}$  there is a standard natural number  $m$  such that  $\mathbf{d}(x_1, x_2) \leq m$ .

The **local ultraproduct** of  $\mathbf{M}_{i \in I}$  is the  $L^+$ -structure  $\mathbf{M}_{\text{loc}}$  obtained as the quotient

$$\mathbf{M}_{\text{loc}} := {}^*\mathbf{M}_{\text{loc}} / \approx$$

where

$$x_1 \approx x_2 \Leftrightarrow \forall n \in \mathbb{N} \mathbf{d}(x_1, x_2) \leq \frac{1}{n}.$$

Since (4) - (9) holds in  $\mathbf{M}_{\text{loc}}$ . We obtain:

**2.6 Proposition.**  $\mathbf{M}_{\text{loc}}$  is a complete metric structure with  $L$ -predicates defining closed subsets and  $L$ -function symbols defining continuous operations on the metric space.

There is an  $L$ -homomorphism

$$\text{lm}^{\text{loc}} : {}^*\mathbf{M}_{\text{loc}} \rightarrow \mathbf{M}_{\text{loc}} \quad (11)$$

onto a metric structure  $\mathbf{M}_{\text{loc}}$  which is the standard part map on values of distance.

**2.7** Accordingly, in analogy with (2), we write

$$\text{lm}^{\text{loc}} \mathbf{M}_i = \mathbf{M}_{\text{loc}} \quad (12)$$

local approximation by  $\mathbf{M}_{i \in I}$ , determined by a metric and the standard part map.

**2.8 Remark.** Note that  ${}^*\mathbf{M}_{\text{loc}}$  is a substructure of  ${}^*\mathbf{M}$  from (1). If we also assume, as is often the case, that there is an embedding

$$\mathbf{M}_{\text{loc}} \hookrightarrow \mathbf{M} \tag{13}$$

then the map  $\text{lm}^{\text{loc}}$  in (11) is a partial homomorphism  ${}^*\mathbf{M} \rightarrow \mathbf{M}$  and so, according to the general theory in [Weg66] and [Z14], it can be extended to a global homomorphism  $\text{lm}$  as in (2).

This proves the following statement.

**Theorem.** *Assume that  $\mathbf{M}$  is quasi-compact  $T_1$ -topological structure and suppose (11) and (12) for some  $\mathbf{M}_{i \in I}$ . Then the map  $\text{lm}^{\text{loc}}$  can be extended to  $\text{lm}^{\text{glob}}$  as in (1).*

*Thus, a local approximation of topological structures could be extended to a global one.*

**2.9 Approximation of metric structures.** Suppose now that  $L = L^+$  is a vocabulary of an unbounded metric structure (continuous logic) with sorts  $S_m$  of diameter  $m$ , and  $\mathbf{M}_i$  are  $L^+$ -metric structures. Then the metric ultraproduct as defined in [BBHU08] is the same as

$$\prod_{\mathcal{D}}^{\text{loc}} \mathbf{M}_i / \approx$$

that is isomorphic to  $\mathbf{M}_{\text{loc}}$ .

*The ultraproduct of metric structures is a local approximation.*

**2.10 Theorem.** *Let, for each  $i \in I$ ,  $\sigma_{a(i)}$  be a general structure formula in vocabulary  $L^+$  with parameter  $a(i) \in \mathbf{M}_i$  and quantifiers restricted to  $S_m$ . Then*

$$\text{lm}_{\mathcal{D}} \sigma_{a(i)}^{\mathbf{M}_i} = \sigma_a^{\mathbf{M}_{\text{loc}}}, \quad a = \text{lm}_{\mathcal{D}} a(i)$$

where  $\text{lm}_{\mathcal{D}}$  is the ultraproduct limit of  $\mathbf{M}_i$ ,  $i \in I$  along the ultrafilter  $\mathcal{D}$  in the sense of general structures of Kiesler.

**Proof.** This is a direct consequence of [Kies20], 2.2.2.  $\square$

### 3 Fields and rings

**3.1** We may assume, using an argument in [Z25-1], that non-standard numbers  $\mathcal{N}, \mathfrak{q} \in {}^*\mathbb{Z}_{>0}$  are such that there is a model  ${}^f\mathbb{Z}$  of arithmetic

$$\mathbb{Z} \prec {}^f\mathbb{Z} \prec {}^*\mathbb{Z} \text{ and } \mathcal{N} \geq \mathfrak{q} > {}^f\mathbb{Z} \tag{14}$$

( ${}^f\mathbb{Z}$  is the “feasible” part of  ${}^*\mathbb{Z}$ ).

Let  $\mathfrak{l} \in {}^f\mathbb{Z}_{>0}$  be an infinite integer (a feasible unit). Note that by (14)  $\mathfrak{l}^n < \mathfrak{q}$  for all  $n \in \mathbb{N}$ .

Set

$${}^f\mathbb{Z}_{/\mathfrak{l}}(n) := \{k \in {}^*\mathbb{Z} : |k| < \mathfrak{l}^n\}, \quad {}^f\mathbb{Z}_{/\mathfrak{l}} := \bigcup_{n \in \mathbb{N}} {}^f\mathbb{Z}_{/\mathfrak{l}}(n).$$

**3.2** Let

$$F = {}^*\mathbb{Z}_{\mathfrak{q}} \cong \prod_{\mathcal{D}} F_q$$

be a pseudofinite field, an ultraproduct of finite prime fields  $F_q$ , where  $\mathfrak{q} \in {}^*\mathbb{Z}$  is the non-standard prime represented by finite primes  $q \in \mathbb{Z}$  along the non-principal ultrafilter  $\mathcal{D}$  on  $\mathbb{N}$ . We may assume that  $\text{Primes} \in \mathcal{D}$  and  $\mathfrak{q}$  is represented in  $q$ -th coordinate of the ultraproduct by  $q$ , that is  $\mathfrak{q}(q) = q$ .

By assumptions, there is an injective map, for all  $n$ ,

$$i_F : {}^f\mathbb{Z}_{/\mathfrak{l}}(n) \subset F; \quad k \mapsto k \bmod \mathfrak{q} \quad (15)$$

and we will assume that it is actual embedding.

Consider also the subsets of  $F$ , for  $m \in \mathbb{N}$ ,

$$S_m(F) := \{z \in F : \exists k_1, k_2 \in {}^f\mathbb{Z}_{/\mathfrak{l}}(m) \ z = k_1 \cdot k_2^{-1} \ \& \ \frac{|k_1|}{|k_2|} \leq m\}$$

$$F_{/\mathfrak{l}} := \bigcup_{m \in \mathbb{N}} S_m(F). \quad (16)$$

**Claim.**  $F_{/\mathfrak{l}}$  is a subfield of  $F$ .

Indeed,  $z_1, z_2 \in S_m(F)$  implies  $z_1 \cdot z_2 \in S_{m^2}(F)$  and  $z_1 + z_2 \in S_{2m}(F)$ .

**3.3** For an element  $z = z(k_1, k_2) \in S_m(F)$  as above define

$$\|z\| := \text{st}\left(\frac{|k_1|}{|k_2|}\right) \in \mathbb{R}_{\geq 0}$$

where  $k_1, k_2$  is the minimal pair such that  $z = k_1 \cdot k_2^{-1}$ .

This is a  $\mathbb{R}$ -valued norm on the subfield  $F_{/\mathfrak{l}}$  of  $F$ . Define respectively

$$d(z_1, z_2) = \|z_1 - z_2\|.$$

The same calculation as in the proof of the claim in 3.2 show that the norm and the distance are well-defined for the infinite pseudo-finite  $F$  with  $\mathfrak{l}$  as in 3.1.

**3.4 Finite fields.** By definitions the element  $\mathfrak{l} \in {}^*\mathbb{Z}$  is represented in each coordinate of the ultrafilter by  $\mathfrak{l}(q) \in \mathbb{Z}$ , the sequence  $\mathfrak{l}(q)$  is unique modulo the ultrafilter.

For each  $n \in \mathbb{N}$  the embedding (15) holds for almost all  $q$  with respect to  $\mathcal{D}$  and we can formally apply the above definitions to  $F = F_q$  interpreting  $\mathfrak{l}$  as  $\mathfrak{l}(q)$ . In this setting the union (16) is equivalent to a finite union and  $F_{/\mathfrak{l}} = F$ . Note that the distance satisfies (4) only for  $x_1, x_2 \in S_m(F)$  with  $\mathfrak{l}(q)^m$  sufficiently small compared to  $q$ .

**Proposition.** *Finite fields  $F_q$  locally approximate the field of real numbers:*

$$\text{lm}^{\text{loc}}F_q = \mathbb{R}. \quad (17)$$

**Proof.** The local ultraproduct of finite fields by the construction is the image of  $F_{/\mathfrak{l}}$ , which consists of elements  $k_1 \cdot k_2^{-1}$  which represent non-standard finite rational numbers embedded into the first-order ultraproduct of fields. Factoring by the equivalence  $\approx$  corresponds to application of the standard part map, which gives us  $\mathbb{R}$ .  $\square$

**Remark.** In [Z14], Theorem 5.2 states that the only structure globally approximated by finite fields  $F_q$  is the compactification  $\bar{E}$  of an algebraically closed field  $E$  of characteristic 0. And if  $E$  is a field with metric, then  $E = \mathbb{C}$ , the field of complex numbers. Thus, the local approximation (17) can be extended to a global one

$$\text{lm}^{\text{glob}}F_q = \bar{\mathbb{C}} \quad (18)$$

and the limit object here is unique up to isomorphism.

Note that the construction of  $\text{lm}^{\text{glob}}$  is different from  $\text{lm}$  in [Z25-1] as the latter uses a different emergent metric on  $F_q$ .

**3.5 Rings.** Consider a pseudo-finite ring

$$K = K_{\mathcal{N}} = {}^*\mathbb{Z}_{\mathcal{N}} \cong \prod \mathbb{Z}_n,$$

the ultraproduct of finite residue rings  $\mathbb{Z}_n$ , as topological structure in Zariski topology.

We will assume that the  $\mathfrak{q}|\mathcal{N}$  and thus there is a surjective ring homomorphism

$$s_{K,F} : K \rightarrow F; \quad k \mapsto k \bmod \mathfrak{q}.$$

On the other hand, like in (15) there an embedding of rings

$$i_K : {}^f\mathbb{Z}_{/\mathfrak{l}} \hookrightarrow K$$



and this agrees with  $i_F$  in such a way that  $s_{K,F} \circ i_K = i_F$ .

Define

$$K_{/I} := s_{K,F}^{-1} F_{/I}$$

and use pseudo-metric  $\mathbf{d} = \mathbf{d}_F$  on  $F_{/I}$  to define a pseudo-metric on  $K_{/I}$

$$\mathbf{d}_K(x, y) := \mathbf{d}_F(s_{K,F}(x), s_{K,F}(y)).$$

Set, for  $m \in \mathbb{N}$ ,

$$K_{/I}(m) := \{z \in K_{/I} : \mathbf{d}(0, z) \leq m\}.$$

With these notations we can proceed with the same construction and arguments as in the case of fields and obtain

$$\mathbf{lm}^{\text{loc}} K_n = \mathbb{R}. \tag{19}$$

In the global setting we obtain again

$$\mathbf{lm}^{\text{glob}} K_n = \bar{\mathbb{C}} \tag{20}$$

due to the fact that  $\mathbf{lm}^{\text{loc}}$  filters through  $K \rightarrow F$ ,

**3.6** Note that (19) is written [Z25-2] as  $\text{Lm } K_{\text{Loc}} = \mathbb{R}$  and was used therein to prove Theorem 3.7 (19), which in the current notation is

$$\mathbf{lm}^{\text{loc}} (\mathbf{M}(K_n), \text{SL}(2, K_n^{(2)})) = (\mathbf{M}(\mathbb{R}), \text{SL}(2, \mathbb{C})).$$

Recalling that the 2-element centre  $C_2$  of  $\text{SL}(2, \mathbb{C})$  acts on  $\mathbf{M}(\mathbb{R})$  trivially and that  $\text{SL}(2, \mathbb{C})/C_2 \cong \text{SO}(1, 3)$ , we establish the equivalent statement in terms of the Lorentz group  $\text{SO}(1, 3)$  :

$$\mathbf{lm}^{\text{loc}} (\mathbf{M}(K_n), \text{SL}(2, K_n^{(2)})/C_2) = (\mathbf{M}(\mathbb{R}), \text{SO}^+(1, 3)).$$

that is Minkowski space equipped with Minkowski metric and the invariant action of Lorentz group on it is locally approximable by finite spaces  $\mathbf{M}(K_n)$  equipped with an analogue of Minkowski metric with invariant actions of finite groups  $\text{SL}(2, K_n^{(2)})/C_2$ .

**3.7 Varieties over fields and rings.** Let  $V \subseteq A^n$  be an affine variety defined over  $\mathbb{Z}$ ,  $V(F_q)$  its set of  $F_q$ -points. We want to calculate  $\text{lm}^{\text{loc}}V(F_q)$ , where  $\text{lm}^{\text{loc}}$  on  $F_q$ -points is the same as in (17). In other words, the structure in question is an affine variety over a field and emerging metric is determined on the fields in 3.2 - 3.4.

**Proposition.**

$$\text{lm}^{\text{loc}}V(F_q) = \overline{V(\mathbb{Q})} \subseteq V(\mathbb{R})$$

*the set of limit points of the set of  $\mathbb{Q}$ -points in  $V(\mathbb{R})$ .*

**Proof.** For the pseudofinite  $F_q = F$  we need to consider the image of  $V(F_{/l})$ . The argument in the proof of 3.4 tells us that this is the same as the image of  $V(*\mathbb{Q}_{\text{fin}})$  under the standard part map, which gives us  $\overline{V(\mathbb{Q})}$ .  $\square$

**3.8 Corollary.** With the same notation, for finite rings  $K_n$  as above,

$$\text{lm}^{\text{loc}}V(K_n) = \overline{V(\mathbb{Q})} \subseteq V(\mathbb{R}).$$

**3.9 .** It was established in [Z25-2] and above in that the Lorentz group  $\text{SO}^+(1, 3)$  is locally approximated by finite groups.

## 4 Compact simple Lie groups

**4.1** We first consider some well-known cases.

A.  $\text{SO}_n(\mathbb{R})$ ,  $n > 2$ . It is well-known, see [Borel91], 18.2, that rational points form a dense subgroup  $\text{SO}_n(\mathbb{Q})$  in each such group. Using 3.7 we get local approximation by finite algebraic groups

$$\text{lm}^{\text{loc}}\text{SO}_n(F_q) = \text{SO}_n(\mathbb{R}).$$

B.  $\text{SU}_n(\mathbb{C})$ ,  $n \geq 2$ , It follows from the general theory, see [Pl-Rap93], Chapter 6, that  $\text{SU}_n$  is rational over  $\mathbb{Q}[i]$  and so  $\text{SU}_n(\mathbb{Q}[i])$  is dense in  $\text{SU}_n(\mathbb{C})$ .

Consider the rings  $F_q^{(2)} = F_q \times F_q$  with operations

$$\begin{aligned} (a_1, b_1) + (a_2, b_2) &= (a_1 + a_2, b_1 + b_2), \\ (a_1, b_1) \cdot (a_2, b_2) &= (a_1 \cdot a_2 - b_1 \cdot b_2, a_1 \cdot b_2 + b_1 \cdot a_2) \end{aligned}$$

It is clear that, using the same emerging metric on  $F_q$ ,

$$\text{lm}^{\text{loc}}F_q^{(2)} = \mathbb{R}[i] = \mathbb{C}.$$

and so

$$\text{lm}^{\text{loc}}\text{SU}_n(F_q^{(2)}) = \text{SU}_n(\mathbb{C}).$$

**4.2 Theorem.** *Every simple compact Lie group is locally approximated by finite groups.*

The theorem is in contrast with the fact proved in [NST18] that the global approximation of simple Lie groups by finite groups is not possible.

To prove the theorem we are going through the list of all simple compact Lie groups and establish that each of them satisfies the rationality condition of 3.7.

**4.3 General facts** [Pl-Rap93] Let  $K$  be a number field,  $v$  an archimedean place, and  $G$  an absolutely simple algebraic group over  $K$ .

*Assume  $G(K_v)$  is compact (equivalently:  $G$  is anisotropic over  $K_v$ ). Then  $G(K)$  is dense in  $G(K_v)$  in the real/complex topology.*

Now we apply the theorem to the list of compact simple Lie groups.

#### 4.4 Classical compact groups

1.  $\mathrm{SO}(n)$ ,  $n \geq 3$  (types  $B_n, D_n$ )

$K = \mathbb{Q}$ , archimedean place  $v = \infty$ , so  $K_v = \mathbb{R}$ .

This is the case A of 4.1.

2.  $\mathrm{SU}(n+1)$  (compact type  $A_n$ )

The compact  $\mathrm{SU}(n+1)$  is most naturally realised as a unitary group over an imaginary quadratic field. This is the case B of 4.1 above.

3.  $\mathrm{Sp}(n)$  (type  $C_n$ , the compact symplectic group)

Here one must be a bit more careful: the split symplectic group  $\mathrm{Sp}_{2n}$  over  $\mathbb{Q}$  has noncompact real points  $\mathrm{Sp}_{2n}(\mathbb{R})$ . The compact  $\mathrm{Sp}(n)$  is an inner form, i.e. is representable as the group of isometries of a nondegenerate Hermitian form over the quaternion division algebra of rank  $n$ .

The field  $K = \mathbb{Q}$  and  $G(\mathbb{Q})$  is dense in  $\mathrm{Sp}(n)$ .

4. Exceptional compact groups

For the compact real forms of  $G_2, F_4, E_6, E_7, E_8$ , there are anisotropic  $\mathbb{Q}$ -forms whose real points are exactly the compact groups you listed. One convenient choice in each case is:

Field  $K = \mathbb{Q}$ .

Group: An absolutely simple, simply connected  $\mathbb{Q}$ -group of the given type, anisotropic over  $\mathbb{R}$ .

Concretely:

$G_2$ : Automorphism group of a rational octonion division algebra (a Cayley algebra) over  $\mathbb{Q}$ . Then  $G(\mathbb{R})$  is the compact  $G_2$ .

$F_4$ : Automorphism group of an Albert division algebra over  $\mathbb{Q}$  anisotropic over  $\mathbb{R}$ ;

$E_6, E_7, E_8$ : By Tits's classification of  $\mathbb{Q}$ -forms there exist simply connected  $\mathbb{Q}$ -groups of each type that are anisotropic at  $\infty$ ; their real points are the compact real forms  $G(\mathbb{R})$ .

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