Axioms of Quantum Mechanics in light of Continuous Model Theory

B.Zilber

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Abstract

The aim of this note is to recast somewhat informal axiom system of quantum mechanics used by physicists (Dirac calculus) in the language of Continuous Logic.

We note an analogy between Tarski's notion of cylindric algebras, as a tool of algebraisation of first order logic, and Hilbert spaces which can serve the same purpose for continuous logic of physics.

1 Introduction

1.1 The axiomatic formulation of quantum mechanics was introduced by Paul Dirac in 1930 [1] through a description of Hilbert space, and later developed with greater mathematical rigor in a monograph of 1932 by John von Neumann. Since 1930, Dirac went through several rewritings and new editions to refine his calculus to a level he considered satisfactory. Modern books present Dirac's axioms in a succinct form, often omitting much of the technical detail.

In section 2 we survey the axioms of quantum mechanics following [2]. Readers with a background in logic will notice that what physicists refer to as axioms is very far from what is a conventional set of axioms in a formal language even in its early form as presented e.g. by Hilbert's axiomatisation of geometry [3].

We argue in section 3 that the language that Dirac introduced is that of *continuous logic*. In section 4 we go further and explain that Dirac's axiomatisation has chosen the formalism known to logician as *algebraic logic* as exemplified e.g. by A.Tarski's cylindric algebras representations, see [5]. In fact, Hilbert spaces can be seen as a continuous model theory version of cylindric algebras.

1.2 Continuous logic and continuous model theory were introduced in the monograph [4] in the 1960s and have since been developed and further generalised for various applications, see e.g. [12]. For readers with no background in logic the article of E.Hushovski [8] outlines a philosophy behind the mathematical formalism.

The link between physics formalism and continuous logic was proposed and initially explored by the present author in [6]. Here this relationship is studied at a deeper level.

2 Dirac's calculus and axiomatisation of quantum mechanics

Below we reproduce a slighly edited version of axioms from [2], 6.3.

2.1 Axiom 1. The state of a quantum system is described by a vector $|\psi\rangle$ belonging to a complex Hilbert space $\mathcal{H}$. This state is usually called "ket ψ". A complex Hilbert space $\mathcal{H}$ is a vector space, which can be finite dimensional or infinite dimensional, equipped with the complex scalar product (also called inner product) $\langle\psi|\psi'\rangle$ between any pair of states $|\psi\rangle$, $|\psi'\rangle$ in $\mathcal{H}$. The norm, or modulus, of a generic vector $|\psi\rangle \in \mathcal{H}$ is defined as

$$||\psi|| = |\langle\psi|\psi\rangle|$$

and usually $|\psi\rangle$ is normalized to one, i.e.$||\psi|| = 1$. The symbol $\langle\psi|$ which appears in the definition of the norm is called "bra ψ" and it can be intepreted as the fuction

$$\langle\psi| : \mathcal{H} \to \mathbb{C}.$$

For any $|\psi'\rangle \in \mathcal{H}$ this function gives a complex number $\langle\psi|\psi'\rangle$ obtained as scalar product of $|\psi\rangle$ and $|\psi'\rangle$. In a complex Hilbert space $\mathcal{H}$ it exists a set of basis vectors $|\phi_\alpha\rangle$ which are orthonormal, i.e. $\langle\phi_\alpha|\phi_\beta\rangle = \delta(\alpha - \beta)$, and such that

$$|\psi\rangle = \sum_\alpha c_\alpha |\phi_\alpha\rangle \tag{1}$$

for any $|\psi\rangle$, where the coefficients c_α belong to $\mathbb{C}$.

Axiom 2. Any observable (measurable quantity) of a quantum system is described by a self-adjoint linear operator $F : \mathcal{H} \to \mathcal{H}$ acting on the Hilbert space of state vectors.

For any classical observable F it exists a corresponding quantum observable F.

Axioms 3.The possible measurable values of an observable F are its eigenvalues f, such that

$$F|f\rangle = f|f\rangle$$

with $|f\rangle$ the corresponding eigenstate. The observable $|f\rangle$ admits the spectral resolution

$$F = \sum_f f|f\rangle\langle f| \tag{2}$$

where $\{|f\rangle\}$ is the set of orthonormal eigenstates of F, and the mathematical object $\langle f|$, called "bra of f", is a linear map that maps into the complex number. This also satisfy the identity

$$\sum_f |f\rangle\langle f| = \mathrm{I}.$$

Axiom 4. The probability P of finding the state $|\psi\rangle$ in the state $|f\rangle$ (both of norm 1) is given by

$$P = |\langle f|\psi\rangle|^2$$

This probability P is also the probability of measuring the value f of the observable F when the system is in the quantum state $|\psi\rangle$.

Axiom 5. The time evolution of states and observables of a quantum system with Hamiltonian H is determined by the unitary operator

$$K^t := \exp(-i\mathrm{H}t/\hbar)$$

, such that $|\psi(t)\rangle = K^t|\psi\rangle$ is the time-evolved state $|\psi\rangle$.

2.2 Now we make several comments on the axioms.

The term “Hilbert space” here should actually be read as *the rigged Hilbert space* (see [9]) because it differs from the standard definition by accommodating both a Hilbert space Φ and the dual space Φ^* with

$$\Phi \subseteq \mathcal{H} \subseteq \Phi^*.$$

The summation formulas like (1) and (2) are presented in a form of an integral if the family $|\psi_\alpha\rangle$ is continuous but seems natural in the summation form when α runs in the discrete spectrum of an operator.

2.3 Remark. Rigged Hilbert spaces provide a powerful mathematical framework to extend quantum mechanics, allowing distributions and generalized eigenfunctions to be rigorously handled. However, as is almost generally accepted, not every element corresponds to a physically realisable state – some are purely mathematical artifacts, see e.g. [10].

In the more general context of quantum field theories Wightman axioms explicitly postulate that physically meaningful part of the rigged Hilbert space $\mathcal{H}$ is a dense subset $\mathcal{D} \subset \mathcal{H}$.

3 The axioms in the setting of Continuous Logic

3.1 We discuss in the section the possible interpretation of the above axioms in terms of (a most general versions of) continuous logic (CL).

Recall that historical prototype of a ket-vector $|\psi\rangle$ of the Hilbert space was a wave-function, that is a function

$$\psi : \mathcal{M} \to \mathbb{C}$$

from a configuration space $\mathcal{M}$ into a bounded domain of the complex numbers $\mathbb{C}$.

These can be seen as predicates on a domain which, for simplicity, is identified with $\mathbb{R}^n$, where $\mathbb{R}$ is the real line seen as a measure space. Definable predicates of norm 1 will be referred to as states.

We represent

$$\mathbb{R} = \bigcup_{k \in \mathbb{N}} I_k$$

where I_k are intervals of finite length, $I_k \subset I_{k+1}$.

Of special significance are the **momentum and position states**. Momentum states where defined by Dirac as the definable family of predicates of the form

$$|p\rangle := \frac{1}{\sqrt{2\pi}} \mathrm{e}^{ipx}, \quad p \in \mathbb{R}.$$

One can consider a $\mathbb{C}$-linear space generated by the momentum states and define Hermitian inner product, first between the momentum states

$$\langle p_1 | p_2 \rangle := \delta^{\mathrm{Kr}}(p_1 - p_2) \tag{3}$$

where δ^{Kr} is the Kronecker delta (in Dirac's setting this is calculated as the Dirac-delta). This inner product definition can be extended to any pair of elements

$$\psi_1 := \int_{\mathbb{R}} f_1(p)|p\rangle, \;\; \psi_2 := \int_{\mathbb{R}} f_2(p)|p\rangle \tag{4}$$

(understood as $\sum_{p\in\mathbb{R}} f_1(p)|p\rangle$ and $\sum_{p\in\mathbb{R}} f_2(p)|p\rangle$) provided

$$\langle \psi_1 | \psi_2 \rangle := \lim_k \int_{I_k} f_1 \cdot f_2^* \; dp$$

exists and is finite, which is the case for $f_1, f_2 \in L^2(\mathbb{R})$. Note that (4) is also well-defined for f_1 and f_2 of the form e^{iF} where F is a polynomial over $\mathbb{R}$ of degree 2 (one uses the Fresnel integral formula in the general case and has to use the agreement (3) when the leading coefficients in F_1 and F_2 are equal).

The **position** states $|x\rangle$, $x \in \mathbb{R}$, by their physical meaning are characteristic functions

$$|x\rangle := \delta^{\mathrm{Kr}}(x - z)$$

which for convenience of continuous mathematical manipulations have been replaced here and in (3) by Dirac's delta-functions that is by distributions. Equivalently, position states can be represented by linear functionals (bra-vectors)

$$\langle x| : \;\; |\psi\rangle \mapsto \psi(x)$$

or equivalently, for ψ running in $\{|p\rangle \;:\; p \in \mathbb{R}\}$,

$$\langle x| : \;\; |p\rangle \mapsto \frac{1}{\sqrt{2\pi}} \mathrm{e}^{ipx} \tag{5}$$

In model theory terms, the linear functionals $\langle x|$ are imaginary elements in the structure, the interpretation of which is given by (5).

Dirac's calculus in aims at rigorous interpretation of such calculations in terms of finite complex values.

3.2 More generally, let $\mathcal{H}_m$ be the set of all m-ary predicates on $\mathbb{R}$ which by definition has structure of $\mathbb{C}$-vector spaces and

$$\mathbb{C} = \mathcal{H}_0 \subset \ldots \subset \mathcal{H}_m \subset \ldots \subset \mathcal{H}_{m+1} \ldots \mathcal{H}.$$

Also, one uses quantifiers, linear maps written as integrals

$$\phi(z_1, \ldots, z_n) \mapsto \int_{\mathbb{R}} \phi(z_1, \ldots, z_n) dz_n.$$

In fact, this is a collection of linear maps

$$\int : \mathcal{H}_{m+1} \to \mathcal{H}_m,$$

the rules of calculation of which as defined by Dirac [1] improper integration. In particular,

$$\int_{\mathbb{R}} \phi(z_1, \ldots, z_n) dz_n := \lim_{k\to\infty} \int_{I_k} \phi(z_1, \ldots, z_n) dz_n \tag{6}$$

(which fits with the requirements of continuous model theory).

A special binary operation in the spaces, inner product,

$$\mathcal{H}_m \times \mathcal{H}_m \to \mathbb{C}; \quad \langle \phi(z_1, \ldots, z_n), \psi(z_1, \ldots, z_n) \rangle = \int_{\mathbb{R}^m} \phi^* \cdot \psi \, dz_1 \ldots dz_m$$

where ϕ^* is the complex comjugate of ϕ and $\int_{\mathbb{R}^m}$ is m-multiple integral. $\langle \phi | \psi \rangle$ can be seen as a continuous predicate of equality $\phi = \psi$.

One restricts the notion of **state** to predicates ϕ such that $\langle \phi | \phi \rangle = 1$.

An important role in the theory is played by a collection of linear maps (operators)

$$L : \mathcal{H}_m \to \mathcal{H}_m$$

with physical meanings. These can be of the form

$$\phi(\bar{z}_1, \bar{z}_2) \mapsto \int_{\mathbb{R}^k} \alpha(\bar{y}, \bar{z_1}) \cdot \phi(\bar{y}, \bar{z}_2) \; d\bar{y}$$

where $|\bar{y}| = |\bar{z}_1| = k$, $\alpha \in \mathcal{H}_{2k}$, or

$$\phi(x, \bar{z}) \mapsto \frac{\partial \phi(x, \bar{z})}{\partial x} \quad \text{or} \quad \phi(x, \bar{z}) \mapsto x \cdot \phi(x, \bar{z})$$

All of the above together makes the $\mathcal{H}_m$ a collection of Hilbert spaces with linear operators and $\mathcal{H}$ an ambient Hilbert space.

The **time evolution operator** $\exp(-i\mathrm{H}t/\hbar)$ acts on states as a unitary operator determining the evolution of a state in time t with a given Hamiltonian H. A state ϕ_{t_0} determining a system at time t_0 evolves into a state $\phi_t := \exp(-i\mathrm{H}(t - t_0)/\hbar)$ with the *probability amplitude* equal to $\langle \phi_{t_0} | \phi_t \rangle$, which is a complex number of modulus 1. The calculation of the CL-formulae ϕ_t and $\langle \phi_{t_0} | \phi_t \rangle$ (which involve mainly calculations of the application of quantifier $\int$) is the central problem of quantum theory, equivalent to solving the associated Schrödinger equation.

The above (along with further details of the Dirac calculus given in [1]) describes the formulae, the connectives and the quantifiers $\int$ of continuous logic for quantum mechanics.

4 Algebraisation of logic and Hilbert space formalism

4.1 The axiomatic description of quantum mechanical theory in the form of rigged Hilbert space may be quite confusing from the logician point of view – there are no logical sentences which can be called axioms. What Axioms 1 – 5 render instead is the topological-algebraic structure of a Hilbert space with operators.

Recall now the *algebraisation of logic* approach, perhaps less popular among model theorists nowadays, versions of which were introduced by A.Lindenbaum, A.Tarski, P.Halmos for the first order setting.

It is quite natural to see the Hilbert space formalism as the form of algebraic logic in the context of the continuous logic of physics.

The qualification 'physics' seems relevant here because of the specific nature of its predicates (states) and quantifiers.

4.2 Recall that, given a first order structure $\mathcal{A}$ in a language $\mathcal{L}$ one can associate with it the cylindric algebra $\mathfrak{C}\mathcal{A}$ as follows:

Let, for distinct $i_1, \ldots, i_n \in \mathbb{N}$, $F_{i_1,\ldots,i_n}$ be the Lindenbaum algebra of $\mathcal{L}$-formulas in variables $x_{i_1}, \ldots, x_{i_n}$ up to equivalence in $\mathcal{A}$. There is a natural emebedding $F_X \subset F_{X'}$, for sets of variables $X \subset X'$. Respectively one defines the Boolean algebra

$$\mathcal{F} := \bigcup_{i_1,\ldots,i_n} F_{i_1,\ldots,i_n}$$

Now introduce, for each i_k, the quantifier

$$\exists x_{i_k} : F_{X'} \to F_X$$

for each X and X' such that X' differs from X by variable x_{i_k}.

Cylindric algebra $\mathfrak{C}\mathcal{A}$ is the Boolean algebra $\mathcal{F}$ equipped with quantification operators $\exists x_{i_k}$.

The structure $\mathcal{A}$ is an interpretation of $\mathfrak{C}\mathcal{A}$.

4.3 Recall **the basic Theorem on Cylindric Algebras** (see e.g. [5])

Let $\mathcal{A}$ and $\mathcal{B}$ be two structures in the same first-order language, and $\mathfrak{C}\mathcal{A}, \mathfrak{C}\mathcal{B}$ the respective cylindric algebras.

Then $\mathcal{A}$ is elementarily equivalent to $\mathcal{B}$ iff $\mathfrak{C}\mathcal{A} \cong \mathfrak{C}\mathcal{B}$, where the isomorphism identifies sets definable by the same formulas.

4.4 In drawing an analogy with the first order case we intend to treat $\mathcal{H}$ as an analogue of cylindric algebras.

However, the physical theory lacks a clear definition of an **interpretation**, that is of a structure, a model.

The problem of furnishing a definition of a **structure M** and the interpretation of the language of quantum mechanics is in practice practically solved by Dirac for the case of the n-particle quantum mechanics.

One can choose a real manifold $\mathcal{M}$ (in fact, $\mathcal{M} = \mathbb{R}$) for the universe of the structure and set predicates (states, wave functions) to be maps

$$\psi : \mathcal{M}^n \to \mathbb{C}$$

with values in a compact subset of $\mathbb{C}$.

One determines rules of calculating the *rigged Hilbert space* operations over the states, including linear operators (CL-connectives) and the quantifier $\int_{\mathcal{M}} \cdot$.

Write the respective structure as

$$(\mathcal{M}; \mathcal{H}).$$

The fragment of QM where such interpretation is well-defined in the context of continuous logic is the theory of a finite number of *free* particles and more generally "Gaussian" quantum mechanics, determined by a Hamiltonians with quadratic potential (which includes the quantum harmonic oscillator). Such theory, with the choice of operators in $\mathcal{H}$ restricted to unitary operators (the Weyl operators and the time evolution operators) is analysed in [6]. It is noticed that the theory has quantifier elimination under the natural choice of basic predicates. Moreover, the theory has a continuous model $(\mathbb{R}^n; \mathcal{H})$ as well as a family of discrete pseudo-finite models (with the universe $\mathbb{V}_{\mathfrak{v}}$).

To move further from there in the context of [6] one needs to include self-adjoint operators (such as P, Q and H) in the definition of $\mathcal{H}$ along with the operation

$$\exp : L \mapsto \mathrm{e}^{iL}$$

defined for self-adjoint L. Note that interpretation of the time evolution operator $\mathrm{e}^{i\frac{\mathrm{H}}{\hbar}t}$ over $\mathcal{M} = \mathbb{R}$ amounts to a path-integral calculation and requires some non-conventional determination of a non-convergent limit even for the case of the quantum harmonic oscillator, see [7], 7.7.4.

More problems arise with including *perturbation methods* into the formalism. These treat the important Planck constant $\hbar$ as an infinitesimal while physics estimates it by a concrete real number.

Generally, there is a sense of dissatisfuction with Dirac - von Neumann formalism (see [10] for references), more so in the broader setting of quantum field theory.

In the following definition an interpretation is in effect a structure of continuous model theory with operators. Our setting for continuous model theory is not fixed below. It certaily is less restrictive than that in the original [4] and follows [12] in assuming that the universe of a structure is just a measure space.

4.5 Definition. Let $\mathcal{H}$ be a **rigged Hilbert space**,

$$\mathcal{H} = (H; \mathrm{O})$$

where H is a complete Hermitian space, O a collection of linear operators on $H^n \to H^m$, which contains Dirac integrals $\int_{\mathbb{R}^k}$ of (6).

We assume O contains the Weyl operators of the form $\mathrm{e}^{ai\mathrm{P}}$ and $\mathrm{e}^{bi\mathrm{Q}}$ for a, b rational numbers ("rational" Weyl operators).

An $\mathcal{H}$-**structure** is given by

- a universe, or a **configuration space** $\mathbb{V}$, given with a measure μ;

- collections $\mathfrak{F}_n$ $n \in \mathbb{N}$, of **predicates**, that is measurable maps $\psi : \mathbb{V}^n \to \mathbb{C}$, closed under $\mathbb{C}$-linear combinations. This will allow quantifiers in the form of integrals over the measure;

- a Hermitian inner product $\langle \psi_1 | \psi_2 \rangle$; $\mathfrak{F}_n \times \mathfrak{F}_n \to \mathbb{C}$ is defined for all n;

- a collection of linear operators $\mathrm{O}_F = \{L_F : L \in \mathrm{O}\}$:

$$L_F : \mathfrak{F}_n \to \mathfrak{F}_m \text{ for each } L : H^n \to H^m$$

which includes a CL-quantifier $\mathrm{E} : \mathfrak{F}_{n+1} \to \mathfrak{F}_n$, for all $n \in \mathbb{N}$;

- an **interpretation functor** is a homomorphism

$$\mathfrak{C} : \begin{array}{l} \mathfrak{F}_n \to \mathcal{H}^{\otimes n};\ n \in \mathbb{N}, \\ \mathrm{O}_F \twoheadrightarrow \mathrm{O} \end{array}$$

which respects the Hermitian structure and the algebra of linear operators and satisfies the condition:

$$\mathrm{Eig}\, W_F \twoheadrightarrow \mathrm{Eig}\, W$$

surjection between bases of eigenfunctions, for $W \in \mathrm{O}$, $W_F \in \mathrm{O}_F$, Weyl operators;

- $\mathfrak{C} : \mathrm{E} \mapsto \int_{\mathbb{R}}$, the Dirac integration operators $\int_{\mathbb{R}}$ corresponds to CL-quantifier $\mathrm{E} : \mathfrak{F}_{n+1} \to \mathfrak{F}_n$.

The respective continuous model structure structure $V = (\mathbb{V}, \mathfrak{C}, \mathcal{H})$ will be called a **model** of $\mathcal{H}$.

4.6 In general, the images $\mathfrak{C}(\mathfrak{F}_n) \subseteq \mathcal{H}^{\otimes n}$ can be proper subsets of $\mathcal{H}$. This is in agreement with Remark 2.3. Respectively, the class of models of $\mathcal{H}$ corresponds in general to a theory which is not necessarily complete.

4.7 It is not hard to see that the Dirac representation of the space of wave functions on $\mathbb{R}$ as in section 3 provides a "canonical" model for the respective $\mathcal{H}$. However, the general $(\mathbb{V}, \mathfrak{C}, \mathcal{H})$ suggests multitude of other possibilities for models of physical reality.

Conceptually one can think of the functor $\mathfrak{C} : V \mapsto \mathcal{H}$ as a **structural approximation** in the sense of [6] and [11] (called Im therein). $\mathfrak{C}$ approximates a “rough” model of reality V by a “smooth” $\mathcal{H}$. This is indeed how approximation has been applied in [6] to pass from a family of discrete (pseudo-finite) structures to classical Hilbert space setting.

Note, that in [6] we work in a more general setting where states take their values in a discrete (pseudofinite) field F, $\psi : \mathbb{V}^n \to \mathrm{F}$. It is then shown that there is an approximation $\mathsf{lm} : \mathrm{F} \to \mathbb{C}$.

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