A class of quantum Zariski geometries

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1 Introduction

This paper is an attempt to understand the nature of *non-classical* Zariski geometries. Examples of such structures were first discovered in [HZ].

These examples showed that contrary to some expectations, one-dimensional Zariski geometries are not necessarily algebraic curves. Given a smooth algebraic curve C with a big enough group of regular automorphisms, one can produce a "smooth" Zariski curve $\tilde{C}$ along with a finite cover $p : \tilde{C} \to C$. $\tilde{C}$ cannot be identified with any algebraic curve because the construction produces an unusual subgroup of the group of regular automorphisms of $\tilde{C}$ ([HZ], section 10). The main theorem of [HZ] states that every Zariski curve has the form $\tilde{C}$, for some algebraic C. So, only in the limit case, when p is bijective, is the curve algebraic.

A typical example of an unusual subgroup of the automorphism group of such a $\tilde{C}$ is the nilpotent group of two generators $\mathbf{U}$ and $\mathbf{V}$ with the central commutator $\epsilon = [\mathbf{U}, \mathbf{V}]$ of finite order N. So, the defining relations are

$$\mathbf{U}\mathbf{V} = \epsilon\, \mathbf{V}\mathbf{U}, \quad \epsilon^N = 1.$$

This, of course, hints towards the known object of non-commutative geometry, the non-commutative (quantum) torus at the Nth root of unity. This observation encourages us to look for systematic links between non-commutative geometry and model theory. More specifically, we would like to give arguments towards the thesis that any non-classical Zariski geometry is in some way an object of non-commutative (quantum) geometry and the classical ones are just the limit cases of the general situation. Towards this end we carried out some analysis of the above examples in [Z2].

In this paper we attempt to give a general method which associates a "geometric object" to a typical *quantum algebra*. Note that this is in fact an open question. Yu. Manin mentions this foundational problem in [Man] I.1.4. Indeed, in general non-commutative geometry does not assume that one has (as is the case in commutative geometry) a procedure of getting a manifold-like structure from the algebra of "observables", yet it is desirable both for technical and conceptual reasons. See also the survey paper [Sk]. The approach in [RVW] looks quite similar to what our paper suggests.

More specifically, we restrict ourselves with quantum algebras *at roots of unity*.

Strictly speaking the general notion of a *quantum algebra* does not exist, and we have to start our construction by introducing algebraic assumptions on A which make the desired theorem feasible.

The next step, after proving that the geometric object we obtain has the right properties, would be to check if our assumptions cover all interesting cases. If it were the case our assumptions would deserve the status of a definition of a quantum algebra.

Our construction always produces a Zariski geometry and when the algebra in question is big enough the structure is provably non-classical, that is not an object of (commutative) algebraic geometry. This might be seen as a good criterion for the adequacy of the construction. Among the structures which satisfy our assumptions is the quantum group $U_\epsilon(\mathfrak{sl}_2)$, but we couldn't check it for higher-dimensional objects because of algebraic difficulties.

In more detail, we consider F-algebras A over an algebraically closed field F. Our assumptions imply that a typical irreducible A-module is of finite dimension over F.

We introduce the structure associated with A as a two-sorted structure $(\tilde{\mathrm{V}}, \mathrm{F})$ where F is given with the usual field structure and $\tilde{\mathrm{V}}$ is the bundle over an affine variety V of A-modules of a fixed finite F-dimension N. Again by the assumptions the isomorphism types of N-dimensional A-modules are determined by points in V. "Inserting" a module M_m of the corresponding type in each point m of V we get

$$\tilde{\mathrm{V}} = \coprod_{m \in \mathrm{V}} M_m.$$

In fact, for any m belonging to an open subset of V, the module M_m is irreducible.

Our language contains a function symbol $\mathbf{U}_i$ acting on each M_m (and so on the sort $\tilde{\mathrm{V}}$) for each generator $\mathbf{U}_i$ of the algebra A. We also have the binary function symbol for the action of F by scalar multiplication on the modules. Since M_m may be considered an $\mathrm{A}/\mathrm{Ann}\, M_m$-module we have the bundle of finite-dimensional algebras $\mathrm{A}/\mathrm{Ann}\, M_m$, $m \in \mathrm{V}$, represented in $\tilde{\mathrm{V}}$. In typical cases the intersection of all such annihilators is 0. As a consequence of this, the algebra A is faithfully represented by its action on the bundle of modules. In fact the whole construction of the structure is aiming to present the category of all finite dimensional A-modules.

Note that in the case when all the M_m are irreducible our structure is a groupoid in the same sense as in Hrushovski's paper [H]. But in general, e.g. in the case of the quantum group $U_\epsilon(\mathfrak{s}l_2)$ the structure is not a groupoid and this is one of the features that makes it richer and more interesting.

We write down our description of $\tilde{\mathrm{V}}$ as the set of first-order axioms Th(A-mod).

We prove two main theorems.

Theorem A (Sections 2.4 and 3.2) *The theory* Th(A-mod) *is categorical in uncountable cardinals and model complete.*

Theorem B (Section 4.3) $\tilde{\mathrm{V}}$ *is a Zariski geometry in both sorts.*

Theorem A is rather easy to prove, and in fact the proof uses not all of the assumptions on A we assumed. Yet despite the apparent simplicity of the construction, for certain A, $\tilde{\mathrm{V}}$ is not definable in an algebraically closed field, that is $\tilde{\mathrm{V}}(\mathrm{A})$ *is not classical.*

Theorem B requires much more work, mainly the analysis of definable sets. This is due to the fact that the theory of $\tilde{\mathrm{V}}$, unlike the case of Zariski geometries coming from algebraic geometry, does not have quantifier elimination in the natural algebraic language. We hope that this technical analysis will be instrumental in practical applications to noncommutative geometry.

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2 From algebras to structures

2.1 We fix below until the end of the paper an F-algebra A, satisfying the following.

Assumptions.

1. We assume that F is an algebraically closed field and A is an associative unital affine F-algebra with generators $\mathbf{U}_1, \ldots, \mathbf{U}_d$ and defining relations with parameters in a finite $C \subset \mathrm{F}$. We also assume that A is a finite dimensional module over its central subalgebra Z_0.

2. Z_0 is a unital finitely generated commutative F-algebra without zero divisors, so $\operatorname{Max} Z_0$, the space of maximal ideals of Z_0, can be identified with the F-points of an irreducible affine algebraic variety V over C.

3. There is a positive integer N such that to every $m \in \operatorname{Max} Z_0$ we can put in correspondens with m an A-module M_m of dimension N over F with the property that the maximal ideal m annihilates M_m.

   The isomorphism type of the module M_m is determined uniformly by a solution to a system of polynomial equations P^A in variables $t_{ijk} \in \mathrm{F}$ and $m \in \mathrm{V}$ such that:

   for every $m \in \mathrm{V}$ there exists $t = \{t_{ijk} : i \leq d,\ j, k \leq N\}$ satisfying $P^A(t, m) = 0$ and for each such t there is a basis $e(1), \ldots, e(N)$ of the F-vector space on M_m with

   $$\bigwedge_{i \leq d,\ j \leq N} \mathbf{U}_i\, e(j) = \sum_{k=1}^{N} t_{ijk} e(k).$$

   We call any such basis $e(1), \ldots, e(N)$ **canonical**.

4. There is a finite group Γ and a map $g : \mathrm{V} \times \Gamma \to \mathrm{GL}_N(\mathrm{F})$ such that, for each $\gamma \in \Gamma$, the map $g(\cdot, \gamma) : \mathrm{V} \to \mathrm{GL}_N(\mathrm{F})$ is rational C-definable (defined on an open subset of V) and, for any $m \in \mathrm{V}$,

   Dom_m, the domain of definition of the map $g(m, \cdot) : \Gamma \to \mathrm{GL}_N(\mathrm{F})$, is a subgroup of Γ,

   $g(m, \cdot)$ is an injective homomorphism on its domain, and for any two canonical bases $e(1),\dots,e(N)$ and $e'(1),\dots,e'(N)$ of M_m there is $\lambda\in \mathrm{F}^*$ and $\gamma\in \mathrm{Dom}_m$ such that

$$e'(i)=\lambda\sum_{1\le j\le N} g_{ij}(m,\gamma)e(j),\quad i=1,\dots,N.$$

We denote

$$\Gamma_m := g(m,\mathrm{Dom}_m).$$

Remark The correspondence $m\mapsto M_m$ between points in V and the isomorphism types of modules is bijective by the assumption 3. Indeed, for distinct $m_1,m_2\in \mathrm{Max}\,Z_0$ the modules M_{m_1} and M_{m_2} are not isomorphic, for otherwise the module will be annihilated by Z_0.

2.2 The structure

Recall that V(A) or simply V stands for the F-points of the algebraic variety $\mathrm{Max}\,Z_0$. By assumption 2.1.1 this can be viewed as the set of A-modules M_m, $m\in \mathrm{Max}\,Z_0$.

Consider the set $\tilde{\mathrm{V}}$ as the disjoint union

$$\tilde{\mathrm{V}}=\coprod_{m\in \mathrm{V}} M_m.$$

We also pick up arbitrarily for each $m\in\mathrm{V}$ a canonical basis $e=\{e(1),\dots,e(N)\}$ in M_m and all the other canonical bases conjugated to e by Γ_m. We denote the set of bases for each $m\in\mathrm{V}$ as

$$E_m:=\Gamma_m e=\{(e'(1),\dots,e'(N)):\ e'(i)=\sum_{1\le j\le N}\gamma_{ij}e(j),\quad \gamma\in\Gamma_m\}.$$

Consider, along with the sort $\tilde{\mathrm{V}}$ also the field sort F, the sort V identified with the corresponding affine subvariety $V\subseteq \mathrm{F}^k$, some k, and the projection map

$$\pi: x\mapsto m \text{ if } x\in M_m,\ \text{ from } \tilde{\mathrm{V}} \text{ to V}.$$

We assume the *full language of* $\tilde{\mathrm{V}}$ contains:

1. the ternary relation $S(x,y,z)$ which holds if and only if there is $m\in\mathrm{V}$ such that $x,y,z\in M_m$ and $x+y=z$ in the module;

2. the ternary relation $a\cdot x=y$ which for $a\in\mathrm{F}$ and $x,y\in M_m$ is interpreted as the multiplication by the scalar a in the module M_m;

3. the binary relations $\mathbf{U}_i x = y$, $(i = 1, \ldots, d)$ which for $x, y \in M_m$ are interpreted as the actions by the corresponding operators in the module M_m;

4. the relations $E \subseteq \mathrm{V} \times \tilde{\mathrm{V}}^N$ with $E(m, e)$ interpreted as $e \in E_m$.

The *weak language* is the sublanguage of the full one which includes 1-3 above only.

Finally, denote $\tilde{\mathrm{V}}$ the 3-sorted structure $(\tilde{\mathrm{V}}, \mathrm{V}, \mathrm{F})$ described above, with V endowed with the usual Zariski language as the algebraic variety.

Remarks 1.Notice that the sorts V and F are bi-interpretable over C.
2. The map $g : \mathrm{V} \times \Gamma \to \mathrm{GL}_N(\mathrm{F})$ being rational is definable in the weak language of $\tilde{\mathrm{V}}$.

Now we introduce the first order theory Th(A-mod) describing $(\tilde{\mathrm{V}}, \mathrm{V}, \mathrm{F})$. It consists of axioms:

Ax 1. F is an algebraically closed field of characteristic p and V is the Zariski structure on the F-points of the variety $\mathrm{Max}\, Z_0$.

Ax 2. For each $m \in \mathrm{V}$ the action of scalars of F and operators $\mathbf{U}_1, \ldots, \mathbf{U}_d$ defines on $\pi^{-1}(m)$ the structure of an A-module of dimension N.

Ax 3. Assumption 2.1.3 holds for the given P^A.

Ax 4. For the $g : \mathrm{V} \times \Gamma \to \mathrm{GL}_N(\mathrm{F})$ given by the assumption 2.1.4, for any $e, e' \in E_m$ there exists $\gamma \in \Gamma$ such that

$$e'(i) = \sum_{1 \le j \le N} g_{ij}(m, \gamma) e(j), \quad i = 1, \ldots, N.$$

Moreover, E_m is an orbit under the action of Γ_m.

Remark The referee of the paper notes that if M_m is irreducible then associated to a particular collection of coefficients t_{kij} there is a unique (up to scalar multiplication) canonical base for M_m (as in 2.1.3). It follows that the only possible automorphisms of $\tilde{\mathrm{V}}$ which fix all of F are induced by multiplication by scalars in each module (the scalars do not have to be the same for each fibre, and typically are not). So the 'projective' bundle $\coprod_{m\in \mathrm{V}}(M_m/\text{scalars})$ is definable in the field F, but the original $\tilde{\mathrm{V}}$ in general is not (see subsection 2.5).

2.3 Examples We assume below that $\epsilon \in \mathrm{F}$ is a primitive root of 1 of order ℓ, and ℓ is not divisible by the characteristic of F.

1. Let A be generated by $\mathbf{U}, \mathbf{V}, \mathbf{U}^{-1}, \mathbf{V}^{-1}$ satisfying the relations

$$\mathbf{U}\mathbf{U}^{-1} = 1 = \mathbf{V}\mathbf{V}^{-1}, \quad \mathbf{U}\mathbf{V} = \epsilon\mathbf{V}\mathbf{U}.$$

We denote this algebra T^2_ϵ (equivalent to $\mathcal{O}_\epsilon((\mathrm{F}^\times)^2)$ in the notations of [BG]).

The centre $Z = Z_0$ of T^2_ϵ is the subalgebra generated by $\mathbf{U}^\ell, \mathbf{U}^{-\ell}, \mathbf{V}^\ell, \mathbf{V}^{-\ell}$. The variety $\mathrm{Max}\, Z$ is isomorphic to the 2-dimensional torus $\mathrm{F}^* \times \mathrm{F}^*$.

Any irreducible T^2_ϵ-module M is an F-vector space of dimension $N = \ell$. It has a basis $\{e_0, \ldots, e_{\ell-1}\}$ of the space consisting of $\mathbf{U}$-eigenvectors and satisfying, for an eigenvalue μ of $\mathbf{U}$ and an eigenvalue ν of $\mathbf{V}$,

$$\mathbf{U}e_i = \mu\epsilon^i e_i$$
$$\mathbf{V}e_i = \begin{cases} \nu e_{i+1}, & i < \ell - 1, \\ \nu e_0, & i = \ell - 1. \end{cases}$$

We also have a basis of $\mathbf{V}$-eigenvectors $\{g_0, \ldots, g_{\ell-1}\}$ satisfying

$$g_i = e_0 + \epsilon^i e_1 + \cdots + \epsilon^{i(\ell-1)} e_{\ell-1}$$

and so

$$\mathbf{V}g_i = \nu\epsilon^i g_i$$
$$\mathbf{U}g_i = \begin{cases} \mu g_{i+1}, & i < \ell - 1, \\ \mu g_0, & i = \ell - 1. \end{cases}$$

For $\mu^\ell = a \in \mathrm{F}^*$ and $\nu^\ell = b \in \mathrm{F}^*$, $(\mathbf{U}^\ell - a), (\mathbf{V}^\ell - b)$ are generators of $\mathrm{Ann}(M)$. The module is determined uniquely once the values of a and b are given. So, V is isomorphic to the 2-dimensional torus $\mathrm{F}^* \times \mathrm{F}^*$.

The coefficients t_{ijk} in this example are determined by μ and ν, which satisfy the polynomial equations $\mu^\ell = a, \nu^\ell = b$.

$\Gamma_m = \Gamma$ is the fixed nilpotent group of order ℓ^3 generated by the matrices

$$\begin{pmatrix} 0\ 1\ 0 \ldots 0 \\ 0\ 0\ 1 \ldots 0 \\ \ldots\ \ \ldots \\ 1\ 0\ \ldots 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1\ 0\ 0 \ldots 0 \\ 0\ \epsilon\ 0 \ldots 0 \\ \ldots\ \ \ldots \\ 0\ 0\ \ldots \epsilon^{\ell-1} \end{pmatrix}$$

2. Similarly, the d-dimensional quantum torus $T^d_{\epsilon,\theta}$ generated by $\mathbf{U}_1, \ldots, \mathbf{U}_d$, $\mathbf{U}_1^{-1} \ldots, \mathbf{U}_d^{-1}$ satisfying

$$\mathbf{U}_i \mathbf{U}_i^{-1} = 1, \quad \mathbf{U}_i \mathbf{U}_j = \epsilon^{\theta_{ij}} \mathbf{U}_j \mathbf{U}_i, \quad 1 \leq i, j \leq d,$$

where θ is an antisymmetric integer matrix, g.c.d.$\{\theta_{ij} : 1 \leq j \leq d\}) = 1$ for some $i \leq d$.

There is a simple description of the bundle of irreducible modules all of which are of the same dimension $N = \ell$.

$T^d_{\epsilon,\theta}$ satisfies all the assumptions.

3. $\mathrm{A} = U_\epsilon(\mathfrak{sl}_2)$, the quantum universal enveloping algebra of $\mathfrak{sl}_2(\mathrm{F})$. It is given by generators K, K^{-1}, E, F satisfying the defining relations

$$KK^{-1} = 1, \; KEK^{-1} = \epsilon^2 E, \; KFK^{-1} = \epsilon^{-2} F, \; EF - FE = \frac{K - K^{-1}}{\epsilon - \epsilon^{-1}}.$$

The centre Z of $U_\epsilon(\mathfrak{sl}_2)$ is generated by K^ℓ, E^ℓ, F^ℓ and the element

$$C = FE + \frac{K\epsilon + K^{-1}\epsilon^{-1}}{(\epsilon - \epsilon^{-1})^2}.$$

We use [BG], Chapter III.2, to describe $\tilde{\mathrm{V}}$. We assume $\ell \geq 3$ odd.

Let $Z_0 = Z$ and so $\mathrm{V} = \operatorname{Max} Z$ is an algebraic extension of degree ℓ of the commutative affine algebra $K^\ell, K^{-\ell}, E^\ell, F^\ell$.

To every point $m = (a, b, c, d) \in \mathrm{V}$ corresponds the unique, up to isomorphism, module with a canonical basis $e_0, \ldots, e_{\ell-1}$ satisfying

$$\begin{aligned} Ke_i &= \mu \epsilon^{-2i} e_i, \\ Fe_i &= \begin{cases} e_{i+1}, & i < \ell - 1, \\ be_0, & i = \ell - 1, \end{cases} \\ Ee_i &= \begin{cases} \rho e_{\ell-1}, & i = 0, \\ (\rho b + \frac{(\epsilon^i - \epsilon^{-i})(\mu\epsilon^{1-i} - \mu^{-1}\epsilon^{i-1})}{(\epsilon-\epsilon^{-1})^2}) e_{i-1}, & i > 0. \end{cases} \end{aligned}$$

where μ, ρ satisfy the polynomial equations

$$\mu^\ell = a, \qquad \rho b + \frac{\mu\epsilon + \mu^{-1}\epsilon^{-1}}{(\epsilon - \epsilon^{-1})^2} = d \tag{1}$$

and

$$\rho \prod_{i=1}^{\ell-1} \left(\rho b + \frac{(\epsilon^i - \epsilon^{-i})(\mu\epsilon^{1-i} - \mu^{-1}\epsilon^{i-1})}{(\epsilon - \epsilon^{-1})^2} \right) = c. \tag{2}$$

We may characterise V as

$$\mathrm{V} = \{(a, b, c, d) \in \mathrm{F}^4 : \ \exists\ \rho, \mu \ \ (1) \text{ and } (2) \text{ hold } \}.$$

In fact, the map $(a, b, c, d) \mapsto (a, b, c)$ is a cover of the affine variety $A^3 \cap \{a \neq 0\}$ of order ℓ.

In almost all points of V, except for the points of the form $(1, 0, 0, d_+)$ and $(-1, 0, 0, d_-)$, the module is irreducible. In the exceptional cases, for each $i \in \{0, \ldots, \ell - 1\}$ we have exactly one ℓ-dimensional module (denoted $Z_0(\epsilon^i)$ or $Z_0(-\epsilon^i)$ in [BG], depending on the sign) which satisfies the above description with $\mu = \epsilon^i$ or $-\epsilon^i$. The Casimir invariant is

$$d_+ = \frac{\epsilon^{i+1} + \epsilon^{-i-1}}{(\epsilon - \epsilon^{-1})^2} \quad \text{or} \quad d_- = -\frac{\epsilon^{i+1} + \epsilon^{-i-1}}{(\epsilon - \epsilon^{-1})^2}$$

and the module, for $i < \ell - 1$, has the unique proper irreducible submodule of dimension $\ell - i - 1$ spanned by $e(i + 1), \ldots, e(\ell - 1)$. For $i = \ell - 1$ the module is irreducible. According to [BG],III.2 all the irreducible modules of A have been listed above, either as M_m or as submodules of M_m for the exceptional $m \in \mathrm{V}$.

To describe Γ_m consider two canonical bases e and e' in M_m. If e' is not of the form λe, then necessarily $e'_0 = \lambda e_k$, for some $k \leq \ell - 1$, $b \neq 0$ and

$$e'_i = \begin{cases} \lambda e_{i+k}, & 0 \leq i < \ell - k, \\ \lambda b e_{i+k}, & \ell - 1 \geq i \geq \ell - k, \end{cases}$$

If we put $\lambda = \lambda_k = \nu^{-k}$, for $\nu^\ell = b$, we get a finite order transformation. So we can take $\Gamma_{(a,b,c,d)}$, for $b \neq 0$, to be the Abelian group of order ℓ^2 generated by the matrices

$$\begin{pmatrix} 0 \ \nu^{-1} \ 0 \ldots 0 \\ 0 \ 0 \ \nu^{-1} \ldots 0 \\ \ldots \quad \ldots \ \nu^{-1} \\ \nu^{\ell-1} \ 0 \ \ldots 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \epsilon \ 0 \ 0 \ldots 0 \\ 0 \ \epsilon \ 0 \ldots 0 \\ \ldots \quad \ldots \\ 0 \ 0 \ \ldots \epsilon \end{pmatrix}$$

where ν is defined by

$$\nu^\ell = b.$$

When $b = 0$ the group $\Gamma_{(a,0,c,d)}$ is just the cyclic group generated by the scalar matrix with ϵ on the diagonal.

The isomorphism type of the module depends on $\langle a, b, c, d\rangle$ *only.* This basis satisfies all the assumptions 1-4.

$U_\epsilon(\mathfrak{s}l_2)$ is one of the simplest examples of a *quantum group.* Quantum groups, as all bi-algebras, have the following crucial property: *the tensor product* $M_1 \otimes M_2$ *of any two* A*-modules is well-defined and is an* A*-module.* So, the tensor product of two modules in $\tilde{\mathrm{V}}$ produces a $U_\epsilon(\mathfrak{s}l_2)$-module of dimension ℓ^2, definable in the structure, and which 'contains' finitely many modules in $\tilde{\mathrm{V}}$. This defines a multivalued operation on V (or on an open subset of V, in the second case).

More examples and the most general known cases $U_\epsilon(\mathfrak{g})$, for $\mathfrak{g}$ a semisimple complex Lie algebra, and $\mathcal{O}_\epsilon(G)$, the quantised group G, for G a connected simply connected semisimple complex Lie group, are shown to have properties 1 and 2 for the central algebra Z_0 generated by the corresponding U_i^ℓ, $i = 1, \ldots, d$.

The rest of the assumptions are harder to check. We leave this open.

4. $\mathrm{A} = \mathcal{O}_\epsilon(\mathrm{F}^2)$, Manin's quantum plane is given by generators $\mathbf{U}$ and $\mathbf{V}$ and defining relations $\mathbf{UV} = \epsilon\mathbf{VU}$. The centre Z is again generated by $\mathbf{U}^\ell$ and $\mathbf{V}^\ell$ and the maximal ideals of Z in this case are of the form $\langle(\mathbf{U}^\ell - a), (\mathbf{V}^\ell - b)\rangle$ with $\langle a, b\rangle \in \mathrm{F}^2$.

This example, though very easy to understand algebraically, does not quite fit into our construction. Namely, the assumption 3 is satisfied only in generic points of $\mathrm{V} = \mathrm{Max}\, Z$. But the main statement still hold true for this case as well. We just have to construct $\tilde{\mathrm{V}}$ by glueing two Zariski spaces each corresponding to a localisation of the algebra A.

To each maximal ideal with $a \neq 0$ we put in correspondence the module of dimension ℓ given in a basis $e_0, \ldots, e_{\ell-1}$ by

$$\mathbf{U}e_i = \mu\epsilon^i e_i$$
$$\mathbf{V}e_i = \begin{cases} e_{i+1}, & i < \ell - 1, \\ be_0, & i = \ell - 1. \end{cases}$$

for μ satisfying $\mu^\ell = a$.

To each maximal ideal with $b \neq 0$ we put in correspondence the module of dimension ℓ given in a basis $g_0, \dots, g_{\ell-1}$ by

$$\mathbf{V} g_i = \nu \epsilon^i g_i$$
$$\mathbf{U} g_i = \begin{cases} g_{i+1}, & i < \ell - 1, \\ a e_0, & i = \ell - 1. \end{cases}$$

for ν satisfying $\nu^\ell = b$.

When both $a \neq 0$ and $b \neq 0$ we identify the two representations of the same module by choosing g (given e and ν) so that

$$g_i = e_0 + \nu^{-1}\epsilon^i e_1 + \cdots + \nu^{-k}\epsilon^{ik} e_k + \cdots + \nu^{-(\ell-1)}\epsilon^{i(\ell-1)} e_{\ell-1}.$$

This induces a definable isomorphism between modules and defines a glueing between $\tilde{\mathrm{V}}_{a\neq 0}$ and $\tilde{\mathrm{V}}_{b\neq 0}$. In fact $\tilde{\mathrm{V}}_{a\neq 0}$ corresponds to the algebra given by three generators $\mathbf{U}, \mathbf{U}^{-1}$ and $\mathbf{V}$ with relations $\mathbf{UV} = \epsilon \mathbf{VU}$ and $\mathbf{U}\mathbf{U}^{-1} = 1$, a localisation of $\mathcal{O}_\epsilon(\mathrm{F}^2)$, and $\tilde{\mathrm{V}}_{b\neq 0}$ corresponds to the localisation by $\mathbf{V}^{-1}$.

2.4 Categoricity

Lemma (i) Let $\tilde{\mathrm{V}}_1$ and $\tilde{\mathrm{V}}_2$ be two structures in the weak language satisfying 2.1.1-2.1.3 and 2.2.1-2.2.3 with the same P^A over the same algebraically closed field F. Then the natural isomorphism $i : \mathrm{V}_1 \cup \mathrm{F} \to \mathrm{V}_2 \cup \mathrm{F}$ over C can be lifted to an isomorphism

$$i : \tilde{\mathrm{V}}_1 \to \tilde{\mathrm{V}}_2.$$

(ii) Let $\tilde{\mathrm{V}}_1$ and $\tilde{\mathrm{V}}_2$ be two structures in the full language satisfying 2.1.1-2.1.4 and 2.2.1-2.2.4 with the same P^A over the same algebraically closed field F. Then the natural isomorphism $i : \mathrm{V}_1 \cup \mathrm{F} \to \mathrm{V}_2 \cup \mathrm{F}$ over C can be lifted to an isomorphism

$$i : \tilde{\mathrm{V}}_1 \to \tilde{\mathrm{V}}_2.$$

Proof We may assume that i is the identity on V and on the sort F.

The assumptions 2.1 and the description 2.2 imply that in both structures $\pi^{-1}(m)$, for $m \in \mathrm{V}$, has the structure of a module. Denote these $\pi_1^{-1}(m)$ and $\pi_2^{-1}(m)$ in the first and second structure correspondingly.

For each $m \in \mathrm{V}$ the modules $\pi_1^{-1}(m)$ and $\pi_2^{-1}(m)$ are isomorphic.

Indeed, using 2.1.3 choose t_{ijk} satisfying P^A for m and find bases e in $\pi_1^{-1}(m)$ and e' in $\pi_2^{-1}(m)$ with the $\mathbf{U}_i$'s represented by the matrices $\{t_{ijk} : k, j = 1, \ldots, N\}$ in both modules. It follows that the map

$$i_m : \sum z_j e(j) \mapsto \sum z_j e'(j), \quad z_1, \ldots, z_N \in \mathrm{F}$$

is an isomorphism of the A-modules

$$i_m : \pi_1^{-1}(m) \to \pi_2^{-1}(m).$$

Hence, the union

$$\mathbf{i} = \bigcup_{m \in \mathrm{V}} i_m, \quad \mathbf{i} : \tilde{\mathrm{V}}_1 \to \tilde{\mathrm{V}}_2,$$

is an isomorphism. This proves (i).

In order to prove (ii) choose, using 2.1.4, e and e' in E_m in $\pi_1^{-1}(m)$ and $\pi_2^{-1}(m)$ correspondingly. Then the map i_m by the same assumption also preserves E_m, and so $\mathbf{i}$ is an isomorphism in the full language. $\square$

As an immediate corollary we get

Theorem Th(A-mod) *is categorical in uncountable cardinals both in the full and the weak languages.*

Remark 1 The above Lemma is a special case of Lemma 3.2.

Remark 2 It is not difficult to see that in the general case the theory Th(A-mod) is not almost strongly minimal in the weak language but is always almost strongly minimal in the full language.

2.5 We prove in this subsection that despite the simplicity of the construction and the proof of categoricity the structures obtained from algebras A in our list of examples are nonclassical.

Assume for simplicity that $\mathrm{char}\, \mathrm{F} = 0$. The statements in this subsection are in their strongest form when we choose the weak language for the structures.

Proposition $\tilde{\mathrm{V}}(T^n_\epsilon)$ *is not definable in an algebraically closed field, for* $n \geq 2$.

Proof We write A for T^2_ϵ. We consider the structure in the weak language.

Suppose towards the contradiction that $\tilde{\mathrm{V}}(\mathrm{A})$ is definable in some F'. Then F is also definable in this algebraically closed field. But, as is well-known, the only infinite field definable in an algebraically closed field is the field itself. So, $\mathrm{F}' = \mathrm{F}$ and so we have to assume that $\tilde{\mathrm{V}}$ is definable in F.

Given $\mathbf{W} \in \mathrm{A}$, $v \in \tilde{\mathrm{V}}$, $x \in \mathrm{F}$ and $m \in \mathrm{V}$, denote $\mathrm{Eig}(\mathbf{W}; v, x, m)$ the statement:

v is an eigenvector of $\mathbf{W}$ *in* $\pi^{-1}(m)$ *(or simply in M_m) with the eigenvalue* x.

For any given $\mathbf{W}$ the ternary relation $\mathrm{Eig}(\mathbf{W}; v, x, m)$ is definable in $\tilde{\mathrm{V}}$ by 2.2.

Let $m \in \mathrm{V}$ be such that μ is an $\mathbf{U}$-eigenvalue and ν is a $\mathbf{V}$-eigenvalue in the module M_m. $\langle \mu^\ell, \nu^\ell \rangle$ determines the isomorphism type of M_m (see 2.3), in fact $m = \langle \mu^\ell, \nu^\ell \rangle$.

Consider the definable set

$$\mathrm{Eig}(\mathbf{U}) = \{v \in \tilde{\mathrm{V}} : \ \exists \mu, m \mathrm{Eig}(\mathbf{U}; v, \mu, m)\}.$$

By our assumption and elimination of imaginaries in ACF this is in a definable bijection with an algebraic subset S of F^n, some n, defined over some finite C'. We may assume that $C' = C$. Moreover the relations and functions induced from $\tilde{\mathrm{V}}$ on $\mathrm{Eig}(\mathbf{U})$ are algebraic relations definable in F over C.

Consider μ and ν as variables running in F and let $\tilde{\mathrm{F}} = \mathrm{F}\{\mu, \nu\}$ be the field of Puiseux series in variables μ, ν. Since $S(\tilde{\mathrm{F}})$ as a structure is an elementary extension of $\mathrm{Eig}(\mathbf{U})$ there is a tuple, say e_μ, in $S(\tilde{\mathrm{F}})$ which is an $\mathbf{U}$-eigenvector with the eigenvalue μ.

By definition the coordinates of e_μ are Laurent series in the variables $\mu^{\frac{1}{k}}$ and $\nu^{\frac{1}{k}}$, for some positive integer k. Let K be the subfield of $\tilde{\mathrm{F}}$ consisting of all Laurent series in variables $\mu^{\frac{1}{k}}$, $\nu^{\frac{1}{k}}$, for the k above. Fix $\delta \in \mathrm{F}$ such that

$$\delta^k = \epsilon.$$

The maps

$$\xi : t(\mu^{\frac{1}{k}}, \nu^{\frac{1}{k}}) \mapsto t(\delta\mu^{\frac{1}{k}}, \nu^{\frac{1}{k}}) \text{ and } \zeta : t(\mu^{\frac{1}{k}}, \nu^{\frac{1}{k}}) \mapsto t(\mu^{\frac{1}{k}}, \delta\nu^{\frac{1}{k}}),$$

for $t(\mu^{\frac{1}{k}}, \nu^{\frac{1}{k}})$ Laurent series in the corresponding variables, obviously are automorphisms of K over F. In particular ξ maps μ to $\epsilon\mu$ and leaves ν fixed, and ζ maps ν to $\epsilon\nu$ and leaves μ fixed. Also note that the two automorphisms commute and both are of order ℓk.

Since $\mathbf{U}$ is F-definable, $\xi^m(e_\mu)$ is a $\mathbf{U}$-eigenvector with the eigenvalue $\epsilon^m\mu$, for any integer m.

By the properties of A-modules $\mathbf{V}e_\mu$ is an $\mathbf{U}$-eigenvector with the eigenvalue $\epsilon\mu$, so there is $\alpha \in \tilde{\mathrm{F}}$

$$\mathbf{V}e_\mu = \alpha\xi(e_\mu). \tag{3}$$

But α is definable in terms of e_μ, $\xi(e_\mu)$ and C, so by elimination of quantifiers α is a rational function of the coordinates of the elements, hence $\alpha \in K$.

Since $\mathbf{V}$ is definable over F, we have for every automorphism γ of K,

$$\gamma(\mathbf{V}e) = \mathbf{V}\gamma(e).$$

So, (3) implies

$$\mathbf{V}\xi^i e_\mu = \xi^i(\alpha)\xi^{i+1}(e_\mu), \quad i = 0, 1, 2, \dots$$

and, since

$$\mathbf{V}^{k\ell}e_\mu = \nu^{k\ell}e_\mu,$$

applying $\mathbf{V}$ to both sides of (3) $k\ell - 1$ times we get

$$\prod_{i=0}^{k\ell-1} \xi^i(\alpha) = \nu^{k\ell}. \tag{4}$$

Now remember that

$$\alpha = a_0(\nu^{\frac{1}{k}}) \cdot \mu^{\frac{d}{k}} \cdot (1 + a_1(\nu^{\frac{1}{k}})\mu^{\frac{1}{k}} + a_2(\nu^{\frac{1}{k}})\mu^{\frac{2}{k}} + \dots)$$

where $a_0(\nu^{\frac{1}{k}}), a_1(\nu^{\frac{1}{k}}), a_2(\nu^{\frac{1}{k}}) \dots$ are Laurent series in $\nu^{\frac{1}{k}}$ and d an integer. Substituting this into (4) we get

$$\nu^{k\ell} = a_0(\nu^{\frac{1}{k}})^{k\ell}\delta^{\frac{k\ell(k\ell-1)}{2}}\mu^{d\ell} \cdot (1 + a_1'(\nu^{\frac{1}{k}})\mu^{\frac{1}{k}} + a_2'(\nu^{\frac{1}{k}})\mu^{\frac{2}{k}} + \dots)$$

It follows that $d = 0$ and $a_0(\nu^{\frac{1}{k}}) = a_0 \cdot \nu$, for some constant $a_0 \in$ F. That is

$$\alpha = a_0 \cdot \nu \cdot (1 + a_1(\nu^{\frac{1}{k}})\mu^{\frac{1}{k}} + a_2(\nu^{\frac{1}{k}})\mu^{\frac{2}{k}} + \dots) \tag{5}$$

Now we use the fact that $\zeta(e_\mu)$ is an $\mathbf{U}$ eigenvector with the same eigenvalue μ, so by the same argument as above there is $\beta \in K$ such that

$$\zeta(e_\mu) = \beta e_\mu. \tag{6}$$

So,

$$\zeta^{i+1}(e_\mu) = \zeta^i(\beta)\zeta^i(e_\mu)$$

and taking into account that $\zeta^{k\ell} = 1$ we get

$$\prod_{i=0}^{k\ell-1} \zeta^i(\beta) = 1.$$

Again we analyse β as a Laurent series and represent it in the form

$$\beta = b_0(\mu^{\frac{1}{k}}) \cdot \nu^{\frac{d}{k}} \cdot (1 + b_1(\mu^{\frac{1}{k}})\nu^{\frac{1}{k}} + b_2(\mu^{\frac{1}{k}})\nu^{\frac{2}{k}} + \dots)$$

where $b_0(\mu^{\frac{1}{k}}), b_1(\mu^{\frac{1}{k}}), b_2(\mu^{\frac{1}{k}}) \dots$ are Laurent series of $\mu^{\frac{1}{k}}$ and d is an integer.

By an argument similar to the above using (7) we get

$$\beta = b_0 \cdot (1 + b_1(\mu^{\frac{1}{k}})\nu^{\frac{1}{k}} + b_2(\mu^{\frac{1}{k}})\nu^{\frac{2}{k}} + \dots) \tag{7}$$

for some $b_0 \in \mathrm{F}$.

Finally we use the fact that ξ and ζ commute. Applying ζ to (3) we get

$$\mathbf{V}\zeta(e_\mu) = \zeta(\alpha)\zeta\xi(e_\mu) = \zeta(\alpha)\xi\zeta(e_\mu) = \xi(\beta)\zeta(\alpha)\xi(e_\mu).$$

On the other hand

$$\mathbf{V}\zeta(e_\mu) = \beta\mathbf{V}e_\mu = \beta\alpha\xi(e_\mu).$$

That is

$$\frac{\alpha}{\zeta(\alpha)} = \frac{\xi(\beta)}{\beta}.$$

Substituting (5) and (7) and dividing on both sides we get the equality

$$\epsilon^{-1}(1 + a'_1(\nu^{\frac{1}{k}})\mu^{\frac{1}{k}} + a'_2(\nu^{\frac{1}{k}})\mu^{\frac{2}{k}} + \dots) = 1 + b'_1(\mu^{\frac{1}{k}})\nu^{\frac{1}{k}} + b'_2(\mu^{\frac{1}{k}})\nu^{\frac{2}{k}} + \dots$$

Comparing the constant terms on both sides we get the contradiction. This proves the proposition in the case $n = 2$.

To end the proof we just notice that the structure $\tilde{\mathrm{V}}(T^2_\epsilon)$ is definable in any of the other $\tilde{\mathrm{V}}(T^n_\epsilon)$, maybe with a different root of unity. This follows from the fact that the A-modules in all cases have similar description.$\square$

Corollary *The structure* $\tilde{\mathrm{V}}(U_\epsilon(\mathfrak{s}l_2))$ *(Example 2.3.3) is not definable in an algebraically closed field.*

Indeed, consider

$$\mathrm{V}_0 = \{(a,b,c,d) \in \mathrm{V} : b \neq 0,\ c = 0\} \text{ and } \tilde{\mathrm{V}}_0 = \pi^{-1}(\mathrm{V}_0)$$

with the relations induced from $\tilde{\mathrm{V}}$.

Set $\mathbf{U} := K$, $\mathbf{V} = F$ and consider the reduct of the structure $\tilde{\mathrm{V}}_0$ which ignores the operators E and C. This structure is isomorphic to $\tilde{\mathrm{V}}(T^2_{\epsilon^2})$ and is definable in $\mathrm{V}(U_\epsilon(\mathfrak{s}l_2))$, so the latter is not definable in an algebraically closed field.$\square$

Remark Note that T^2_ϵ here does not have any immediate connection to the non-classical Zariski curve T_N in [Z2]. So the Proposition does not "explain" the earlier examples, though an attentive reader could spot similarities in the proof of the Proposition and that of the non-algebraicity of T_N. A possible connection remains an open question.

3 Definable sets

3.1 Given variables $v_{1,1}, \ldots, v_{1,r_1}, \ldots, v_{s,1} \ldots, v_{s,r_s}$ of the sort $\tilde{\mathrm{V}}$, $m_1, \ldots, m_s$ of the sort V and variables $x = \{x_1, \ldots, x_p\}$ of the sort F, denote $A_0(e,m,t)$ the formula

$$\bigwedge_{i\leq s,\ j\leq N} E(e_i, m_i) \ \&\ P^A(\{t_{ikn\ell}\}_{k\leq d,\ \ell,n\leq N}; m_i) = 0 \ \&\bigwedge_{k\leq d, j\leq N, i\leq s} \mathbf{U}_k e_i(j) = \sum_{\ell\leq N} t_{ikj\ell} e_i(\ell).$$

Denote $A(e,m,t,z,v)$ the formula

$$A_0(e,m,t)\ \&\bigwedge_{i\leq s;\ j\leq r_i} v_{ij} = \sum_{\ell\leq N} z_{ij\ell} e_i(\ell).$$

The formula of the form

$$\exists\ e_1, \ldots e_s \exists\ m_1, \ldots, m_s$$

$$\exists \ \{t_{ikjl} : k \le d, \ i \le s, \ j, \ell \le N\} \subseteq \mathrm{F}$$

$$\exists \ \{z_{ijl} : i \le s, \ j \le r_i, \ \ell \le N\} \subseteq \mathrm{F} :$$

$$A(e, m, t, z, v) \& \ R(m, t, x, z),$$

where R is a boolean combination of Zariski closed predicates in the algebraic variety $\mathrm{V}^s \times \mathrm{F}^q$ over C, $q = |t| + |x| + |z|$ (constructible predicate over C) will be called **a core $\exists$-formula with kernel** $R(m, t, x, z)$ **over** C. The enumeration of variables v_{ij} will be referred to as the **partitioning enumeration**.

We also refer to this formula as $\exists e R$.

Comments (i) A core formula is determined by its kernel once the partition of variables (by enumeration) is fixed. The partition sets that $\pi(e_i(j)) = \pi(e_i(k))$, for every i, j, k, and fixes the components of the sub-formula $A(e, m, t, z, v)$.

(ii) The relation $A_0(e, m, t)$ defines the functions

$$e \mapsto (m, t),$$

that is given a canonical basis $\{e_i(1), \ldots, e_i(N)\}$ in M_{m_i} we can uniquely determine m_i and $t_{ikj\ell}$.

For the same reason $A(e, m, t, z, v)$ defines the functions

$$(e, v) \mapsto (m, t, z).$$

3.2 Lemma Let

$$a = \langle a_{1,1}, \ldots, a_{1,r_1}, \ldots, a_{s,1} \ldots, a_{s,r_s} \rangle \in \tilde{\mathrm{V}} \times \cdots \times \tilde{\mathrm{V}}, \ b = \langle b_1, \ldots, b_n \rangle \in \mathrm{F}^n.$$

The complete type $\mathrm{tp}(a, b)$ of the tuple over C is determined by its subtype $\mathrm{ctp}(a, b)$ over C consisting of core $\exists$-formulas.

Proof We are going to prove that, given a', b' satisfying the same core type $\mathrm{ctp}(a, b)$ there is an automorphism of any $\aleph_0$-saturated model, $\alpha : (a, b) \mapsto (a', b')$.

We assume that the enumeration of variables has been arranged so that $\pi(a_{ij}) = \pi(a_{kn})$ if and only if $i = k$. Denote $m_i = \pi(a_{ij})$.

Let e_i be bases of modules $\pi^{-1}(m_i)$, $i = 1, \ldots, s$, $j = 1, \ldots, N$, such that $\models A_0(e, m, t)$ for some $t = \{t_{ikj\ell}\}$ (see the notation in 3.1 and the assumption

2.1.3), in particular $e_i \in E_{m_i}$. By the assumption the correspondent systems span M_{m_i}, so there exist $c_{ij\ell}$ such that

$$\bigwedge_{i\leq s;\ j\leq r_i} a_{ij} = \sum_{\ell\leq N} c_{ij\ell} e_i(\ell),$$

and let $p = \{P_i : i \in \mathbb{N}\}$ be the complete algebraic type of (m, t, b, c).

The type $\mathrm{ctp}(a, b)$ contains core formulas with kernels P_i, $i = 1, 2, \ldots$ By assumptions and saturatedness we can find e' m', t' and c' satisfying the correspondent relations for (a', b'). In particular, the algebraic types of (m, t, b, c) and (m', t', b', c') over C coincide and $e'_i \in E_{m'_i}$. It follows that there is an automorphism $\alpha : \mathrm{F} \to \mathrm{F}$ over C such that $\alpha : (m, t, b, c) \mapsto (m', t', b', c')$.

Extend α to $\pi^{-1}(m_1) \cup \ldots \cup \pi^{-1}(m_s)$ by setting

$$\alpha(\sum_j z_j e_i(j)) = \sum_j \alpha(z_j) e'_i(j) \tag{8}$$

for any $z_1, \ldots, z_N \in \mathrm{F}$ and $i \in \{1, \ldots, s\}$. In particular $\alpha(a_{ij}) = a'_{ij}$ and, since $\alpha(\Gamma_{m_i}) = \Gamma_{m'_i}$, also $\alpha(E_{m_i}) = E_{m'_i}$.

Now, for each $m \in \mathrm{V} \setminus \{m_1, \ldots, m_s\}$ we construct the extension of α, $\alpha^+_m : \pi^{-1}(m) \to \pi^{-1}(m')$, for $m' = \alpha(m)$, as in 2.4. Use 2.1.3 to choose t_{ijk} satisfying P^A for m and find bases $e \in E_m$ and $e' \in E_{m'}$ with the $\mathbf{U}_i$'s represented by the matrices $\{t_{ijk} : k, j = 1, \ldots, N\}$ in $\pi^{-1}(m)$ and by $\{\alpha(t_{ijk}) : k, j = 1, \ldots, N\}$ in $\pi^{-1}(m')$. It follows that the map

$$\alpha^+_m : \sum z_j e(j) \mapsto \sum \alpha(z_j) e'(j), \quad z_1, \ldots, z_N \in \mathrm{F}$$

is an isomorphism of the A-modules

$$\alpha^+_m : \pi^{-1}(m) \to \pi^{-1}(m').$$

Hence, the union

$$\alpha^+ = \bigcup_{m\in\mathrm{V}} \alpha^+_m$$

is an automorphism of $\tilde{\mathrm{V}}$.$\square$

By the compactness theorem we immediately get from the lemma.

Corollary Every formula in $\tilde{\mathrm{V}}$ with parameters in $C \subseteq \mathrm{F}$ is equivalent to the disjunction of a finite collection of core formulas.

3.3 We consider now a more general form of core formulas with parameters in both sorts $\tilde{\mathrm{V}}$ and F.

The **general core formula** of variables $x = \{x_1, \ldots, x_p\}$ and $v = \{v_{ij} : i \leq s+u, j \leq r_i\}$ and parameters $C \subseteq \mathrm{F}$, $\hat{e} \subseteq \tilde{\mathrm{V}}$ will be of the form

$$\exists\ e_1, \ldots e_s \exists\ m_1, \ldots, m_s$$
$$\exists\ \ \{t_{ikjl} : k \leq d,\ i \leq s,\ j, \ell \leq N\}$$
$$\exists\ \{z_{ijl} : i \leq s,\ j \leq r_i,\ \ell \leq N\}$$
$$\exists\ \{y_{ijl} : i \leq u,\ j \leq r_{s+i},\ \ell \leq N\}$$
$$A(e, m, t, z, v) \&\ B(\hat{e}, y, v) \&\ R(m, t, x, y, z),$$

where $\hat{e} = (\hat{e}_{s+1}, \ldots, \hat{e}_{s+u})$ are names of fixed canonical bases of some modules $M_{\hat{m}_{s+1}}, \ldots, M_{\hat{m}_{s+u}}$ in $\tilde{\mathrm{V}}$, m, t, z and A are the same as in 3.1, y is $\{y_{ijl} : i \leq u,\ j \leq r_{s+i},\ \ell \leq N\}$, R is a Boolean combination of Zariski closed predicates in variables m, t, x, y, z and $B(\hat{e}, y, v)$ is the formula

$$\bigwedge_{i \leq u;\ j \leq r_{s+i}} v_{s+i,j} = \sum_{\ell \leq N} y_{ij\ell} \cdot \hat{e}_i(\ell).$$

As before we call R appearing in the general core formula the kernel of the formula and write $\exists e\, R$ for the general core formula with kernel R.

Remark Given the set in $\tilde{\mathrm{V}}$ defined by a general core formula $\exists e\, R$ the values of parameters $\hat{m}_{s+1}, \ldots, \hat{m}_{s+u}$ are determined uniquely as $\pi(v_{ij})$ with $i = s+1, \ldots, s+u$, $j \leq r_i$. Hence $\hat{e}_i$, $s < i \leq s+u$, are determined up to a linear transformation inside $M_{\hat{m}_i}$. So, choosing a different $\hat{e}' = \gamma \hat{e}$ one can still define the same set by using the formula $\exists e\, R'$ where $R'(m, t, x, y', z)$ is obtained from $R(m, t, x, y, z)$ by substituting $y'\gamma$ in place of y. In other words,

we may assume that two equivalent general core formulas have the same parameters $\hat{e}$.

Proposition *Every formula with parameters in* $\tilde{\mathrm{V}}$ *is equivalent to the disjunction of a finite collection of general core formulas.*

Proof By 3.1 it is enough to prove that there is such a form for the formula obtained from a core formula $\exists e\, R$ in variables $x = \{x_1, \dots, x_p\}$ and $v = \{v_{ij} : i \le s+u, j \le r_i\}$ with parameters $C \subseteq \mathrm{F}$,

$$\begin{gathered} \exists\ e_1, \dots e_s, \dots e_{s+u} \exists\ m_1, \dots, m_s, \dots, m_{s+u} \\ \exists\ \ \{t_{ikjl} : i \le s+u,\ k \le d,\ j, \ell \le N\} \\ \exists\ \{z_{ijl} : i \le s+u,\ j \le r_i,\ \ell \le N\} \\ A(e, m, t, z, v) \&\ R(m, t, x, z) \end{gathered} \tag{9}$$

by substituting

$$v_{s+1,1} := a_{s+1,1}, \dots, v_{s+1,q_{s+1}} := a_{s+1,q_{s+1}},$$

$$\dots$$

$$v_{s+u,1} := a_{s+u,1}, \dots, v_{s+u,q_{s+u}} := a_{s+u,q_{s+u}},$$

some $a_{ij} \in \tilde{\mathrm{V}}$ and $1 \le q_i \le r_i$, $i = s+1, \dots, s+u$.

Notice that once the substitution $v_{s+i,1} := a_{s+i,1}$ occured the value of m_{s+i} will be fixed as $m_{s+i} = \pi(a_{s+i})$. Denote this $\hat{m}_{s+i}$. Correspondingly there are finitely many possible values for $e_{s+i} \in E_{\hat{m}_{s+i}}$. Choosing any such canonical basis $\hat{e}_{s+i}$, the corresponding $t_{s+i,kjl}$ described in $A(e, m, t, z, v)$ will be fixed, denote the correspondent elements in F as $\hat{t}_{s+i,kjl}$. For the same reason we have the $z_{s+i,jl}$, for $j \le q_{s+i}$, fixed as $\hat{z}_{s+i,jl}$ by $A(e, m, t, z, v)$.

So, $\exists e\, R^{v_I := a_I}$ is equivalent to

$$\begin{gathered} \bigvee_{\hat{e}_{s+1} \in E_{\hat{m}_{s+1}} \dots \hat{e}_{s+u} \in E_{\hat{m}_{s+u}}} \exists\ e_1, \dots e_s \exists\ m_1, \dots, m_s \\ \exists\ \ \{t_{ikjl} : i \le s,\ k \le d,\ j, \ell \le N\} \\ \exists\ \{z_{ijl} : i \le s,\ j \le r_i,\ \ell \le N\} \\ (A(e, m, t, z, v) \&\ R(m, t, x, z))^{v_I := a_I, m_I = \hat{m}_I, t_I = \hat{t}_I, z_I = \hat{z}_I} \end{gathered}$$

Now we rename z_{ijl} with $s < i \le s+u$ and $q_i < j \le r_i$ as $y_{i-s, j-q_i, l}$. $R(m, t, x, z)^{v_I := a_I, m_I = \hat{m}_I, t_I = \hat{t}_I, z_I = \hat{z}_I}$ becomes then some constructible predicate in variables m, t, x, y, z and parameters C and $\hat{m}_I, \hat{t}_I, \hat{z}_I$.

We now want to reduce

$$A(e,m,t,z,v)^{v_I:=a_I,m_I=\hat{m}_I,t_I=\hat{t}_I,z_I=\hat{z}_I}$$

to a suitable equivalent form. To this end we delete from the formula the conjuncts which are trivially true, namely $E(\hat{e}_i,\hat{m}_i)$ and the equalities of the form

$$P^A(\{\hat{t}_{ikn\ell}\}_{k\leq d,\ \ell,n\leq N}; m_i)=0, \quad a_{ij}=\sum_l \hat{z}_{ijl}\hat{e}_i(l) \text{ and } \mathbf{U}_k\hat{e}_i(j)=\sum \hat{t}_{ikjl}\hat{e}_i(l)$$

for $i>s$. The only equations with indices $i>s$ remaining will have the form

$$v_{ij}=\sum_l y_{i-s,j,l}\hat{e}_i(l),$$

and the conjunction of all these will form our $B(\hat{e},y,v)$ (we rename $\hat{e}_i$ as $\hat{e}_{i-s}$ in the final form). The remaining part of $A(e,m,t,z,v)^{v_I:=a_I,m_I=\hat{m}_I,t_I=\hat{t}_I,z_I=\hat{z}_I}$ will be exactly $A(e,m,t,z,v)$ where e,m,t,z,v are as in the definition of a general core formula. $\Box$

Remark We have also proved that the result $\exists e\, R^{v_I:=a_I}$ of the substitution in a given core formula (9) with kernel $R(m,t,x,z)$ is equivalent to a disjunction of general core formulas each with the kernel

$$R(m,t,x,z)^{m_I=\hat{m}_I,t_I=\hat{t}_I,z_I=\hat{z}_I y_I},$$

where the substitution $z_I=\hat{z}_I y_I$ replaces z_{ijl} with $s<i\leq s+u$ by $\hat{z}_{ijl}$, for $j\leq q_i$, or by $y_{i-s,j-q_i,l}$, for $q_i<j\leq r_i$.

Corollary Every formula in $\tilde{\mathrm{V}}$ with parameters in $\tilde{\mathrm{V}}$ is equivalent to the disjunction of a finite collection of general core formulas.

3.4 We assume from now on the stronger assumption 2.1.4 and prove in this section that the core formulas in Corollaries of 3.2 and 3.3 can have a form more suitable for technical purposes.

Let Γ be the group in 2.1.4. Given a Zariski closed predicate $R:=R(m,t,x,y,z)$ with m ranging in V^s and t,x,y,z tuples in F in accordance with the notation in 3.2, we define, for $\gamma=(\gamma_1,\ldots,\gamma_s)\in\Gamma^s$, the predicate $R^\gamma(m,t,x,y,z)$ of the same variables.

First we consider the case when R is irreducible. Set

$$\mathrm{V}_R := \{m \in \mathrm{V}^s : \exists t, x, y, z\, R(m, t, x, y, z)\},$$

the projection of R on V^s. Let $\mathrm{V}_{R,\gamma}$ be the open subset of V_R equal to the domain of definition of the map (in variables m)

$$g(m, \gamma) := \langle g(m_1, \gamma_1), \dots, g(m_s, \gamma_s)\rangle.$$

Set

$$\Gamma^s_R := \{\gamma \in \Gamma^s : \mathrm{V}_{R,\gamma} \neq \emptyset\}.$$

This is a subgroup of Γ^s since $\mathrm{V}_{R,\gamma}$ is a dense open subset when nonempty.

In case $\gamma \in \Gamma^s_R$ define R^γ to be the Zariski closure of the set

$$\{\langle m, t, x, y, z\rangle : \exists t', z'\ m \in \mathrm{V}_{R,\gamma}\ \&\ t' = g(m, \gamma)^{-1} \cdot t \cdot g(m, \gamma)\ \&$$
$$\&\ z' = z \cdot g(m, \gamma)\ \&\ R(m, t', x, y, z')\}.$$

Remember that t is a collection of $N \times N$ matrices and z is a list of N-tuples, coordinates of elements of M_{m_i} in the corresponding canonical bases e. So in the definition above $z \cdot g(m, \gamma)$ corresponds to the coordinates of the same elements in bases $e' = g(m, \gamma) \cdot e$, and $g(m, \gamma)^{-1} \cdot t \cdot g(m, \gamma)$ is the result of the corresponding transformation of the matrices in t.

With an obvious abuse of notation we will often write $R(m, t^\gamma, x, y, z\gamma)$ for $R^\gamma(m, t, x, y, z)$, when $\gamma \in \Gamma^s_R$.

For a general Zariski closed R we first represent $R = R_1 \cup \dots \cup R_k$ as the union of its irreducible components and then set

$$R^\gamma = R^\gamma_1 \cup \dots \cup R^\gamma_k,$$

$$\Gamma^s_R = \Gamma^s_{R_1} \cap \dots \cap \Gamma^s_{R_k}.$$

Remarks (i) Obviously, $R^{\mathrm{id}} = R$, for id the unit element of Γ^s;

(ii) For $\gamma \in \Gamma^s_R$ the set $\mathrm{V}_{R^\gamma} \cap \mathrm{V}_R$ is a dense open subset of V_R equal to $\mathrm{V}_R \cap \mathrm{V}_{R,\gamma}$;

(iii) If R does not depend on t and z then $R^\gamma = R$ for every $\gamma \in \Gamma^s_R$;

(iv) Suppose $P \subseteq R$ is a Zariski closed relation. Then $P^\gamma \subseteq R^\gamma$ for every $\gamma \in \Gamma^s$ and $\Gamma^s_P \subseteq \Gamma^s_R$;

(v) Let $R^* = \bigcup_{\gamma \in \Gamma^s} R^\gamma$. Then

$$R^{*\gamma} = R^*$$

for every $\gamma \in \Gamma^s_R$.

We will say that R is Γ-invariant if $R^\gamma = R$ for every $\gamma \in \Gamma^s_R$.

3.5 Lemma 1. *We may assume that the kernels in core formulas in Corollary 3.2 are of the form $R(m,t,x,z)$ & $\neg S(m,t,x,z)$, where R, S are given by systems of equations and S is Γ-invariant.*

Proof We go back to the proof of Lemma 3.2 and consider the deductively complete type p in the language of fields, $p = \{P_i\}$, with conjunctions of P_i appearing in the end as the kernels of core formulas. We may assume that each P_i is either a system of equations $R(m,t,x,z)$ in variables m,t,x,z or the negation $\neg S(m,t,x,z)$ of the system of equations S. We are going to prove that, for a given $\neg S \in p$ there is a system of equations $R \in p$, and a negation $\neg \bar{S} \in p$, with $\neg \bar{S}^\gamma = \neg \bar{S}$, for all $\gamma \in \Gamma^s_S$, such that R & $\neg \bar{S} \models \neg S$. This implies that we can replace all P_i by R & $\neg \bar{S}$ and would prove the Lemma.

If $\bigwedge_{\gamma \in \Gamma^s} \neg S^\gamma \in p$ then this formula, being equivalent to a negation $\neg \bar{S}$ of a system of equations, is invariant under Γ^s and satisfies $\neg \bar{S} \models \neg S$.

So we assume the opposite, $\bigwedge_{\gamma \in \Gamma^s} \neg S^\gamma \notin p$. Hence, for some nonempty proper subset $\Delta \subsetneq \Gamma^s$, with $1 \in \Delta$,

$$\neg T = \bigwedge_{\gamma \in \Delta} \neg S^\gamma \in p.$$

We assume Δ to be maximal with this property.

Obviously

$$\bigvee_{\gamma \in \Gamma^s} \neg T^\gamma \in p.$$

Denote

$$\mathrm{Stab}(\Delta) = \{\gamma \in \Gamma^s : \gamma\Delta = \Delta\}.$$

Since by maximality for any $\gamma \in \Gamma^s \setminus \mathrm{Stab}(\Delta)$ we have $\neg T^\gamma$ & $\neg T \notin p$, necessarily $T^\gamma \in p$ and so

$$\bigwedge_{\gamma \in \Gamma^s \setminus \mathrm{Stab}\Delta} T^\gamma \in p.$$

But

$$\bigvee_{\gamma \in \Gamma^s} \neg T^\gamma \ \& \bigwedge_{\gamma \in \Gamma^s \setminus \mathrm{Stab}\Delta} T^\gamma \models \bigvee_{\gamma \in \mathrm{Stab}\Delta} \neg T^\gamma$$

The latter is equivalent to $\neg T$, and $\neg T \models \neg S$. So we can take $\bigwedge_{\gamma\in\Gamma^s\setminus\mathrm{Stab}\Delta} T^\gamma$ for R and $\bigvee_{\gamma\in\Gamma^s} \neg T^\gamma$ for $\neg \bar{S}$.$\square$

Lemma 2. *We may assume that in Corollary 3.2 the kernels of core formulas are of the form $R\&\neg S$ with R, S given by systems of polynomial equations and both are Γ-invariant.*

Proof We use Lemma 1. Observe first that in general

Claim $\exists e, t, z A(e, m, t, v, z)$ & $P(m, t, z, x)$ is equivalent to $\exists e, t, z A(e, m, t, v, z)$ & $P^*(m, t, z, x)$, where

$$P^* = \bigvee_{\gamma\in\Gamma^s} P^\gamma.$$

Indeed, $\exists e\ P$ obviously implies $\exists e\ P^*$. To see the converse note that, given v and x, if for some e and $\gamma \in \Gamma^s$ we have $\models A(e, m, t, v, z)$ & $P(m, t^\gamma, z\gamma, x)$ then, letting $e' = \gamma e$, we will have $\models A(e', m, t^\gamma, v, z\gamma)$ and so, $\models A(e', m, t', v, z')$ & $P(m, t', z', x)$ for $t' = t^\gamma$ and $z' = z\gamma$.

Applying the Claim to our $\exists e\ R$ & $\neg S$ we will get the equivalent formula $\exists e\ R^*$ & $\neg S$ since $S^* = S$. $\square$

Combining Lemma 2 with the Remark in 3.3 we get.

Corollary *We may assume that in Corollary 3.3 the kernels of general core formulas are of the form $R\&\neg S$ with R, S given by systems of polynomial equations and both are Γ-invariant.*

Now we discuss general core formulas with Γ-invariant kernels.

Lemma 3. *Assuming that R_2 is Γ-invariant we have*
(i) $\exists e\ (R_1\ \&\ R_2) \equiv (\exists e\ R_1)\ \&\ (\exists e\ R_2)$;
(ii) $\exists e\ \neg R_2 \equiv \neg\exists e\ R_2$.

Proof (i) The left-hand-side obviously implies the formula on the right. Assume for converse that the right-hand-side is true. That is for given v, x and y there is e and e' such that $\models A(e, m, t, v, z)\ \&\ B(\hat{e}, y, v)\&\ R_1(m, t, x, y, z)$ and $\models A(e', m, t', v, z')\ \&\ B(\hat{e}, y, v)\&\ R_2(m, t', x, y, z')$. Since $e' = \gamma e$ for some $\gamma \in \Gamma^s$, we have $\models A(e, m, t, v, z)\ \&\ B(\hat{e}, y, v)\&\ R_2(m, t^\gamma, x, y, z\gamma)$. But R_2 is Γ-invariant, hence we get $\models A(e, m, t, v, z)\ \&\ B(\hat{e}, y, v)\&\ R_1(m, t, z, x)\ \&\ R_2(m, t, z, x)$, as requred.

(ii) We need only prove the implication from left to right. Assume that $\models A(e, m, t, v, z)$ & $B(\hat{e}, y, v)$& $\neg R_2(m, t, x, y, z)$. We need to check that for no e' it is possible $\models A(e', m, t', v, z')$ & $B(\hat{e}, y, v)$& $R_2(m, t', x, y, z')$. Indeed, as above by Γ-invariance the latter is equivalent to
$\models A(e, m, t, v, z)$ & $B(\hat{e}, y, v)$& $R_2(m, t, x, y, z)$, which would contradict the former. □

Lemma 4 Suppose $\exists e R_1 \equiv \exists e R_2$, both sides are general core formulas with the same partition of v-variables, u and $\hat{e}_1, \ldots, \hat{e}_u$ are same in both formulas, and R_1, R_2 are Γ-invariant. Then $R_1 \equiv R_2$.

Proof By Lemma 3

$$\exists e\ (R_1\ \&\ \neg R_2) \equiv \exists e\ R_1\ \&\ \exists e\ \neg R_2 \equiv \exists e\ R_1\ \&\ \neg \exists e\ R_2$$

and so R_1 & $\neg R_2$ is inconsistent, that is $\models R_1 \rightarrow R_2$. By symmetry $R_1 \equiv R_2$.□

4 Zariski geometry

In this section we introduce on $\tilde{\mathrm{V}}$ and its finite cartesian powers a topology which is naturally coming from the coordinate algebra A. To see that this is a Noetherean topology satisfying also the definition of a pre-smooth Zariski geometry (see [Z1] for this) we have to have more than just a quantifier elimination to existential formulas. To this end we carry out a more detailed analysis of general core formulas and their behavior under Boolean operations and projections.

4.1 We introduce the A-topology declaring **basic closed subsets** of $\tilde{\mathrm{V}}^n \times \mathrm{F}^p$ the subsets defined by general core formulas $\exists e R$ with kernels R given by Γ-invariant systems of polynomial equations with coefficients in F.

We also assume that R contains the equation $P^A(t, m) = 0$ (see 2.1.3), which is in the A-part of $\exists e R$.

We often denote $\hat{R}$ the closed set defined by the formula $\exists e R$.

The closed subsets of the topology are given by applying finite unions and arbitrary intersections to basic closed subsets.

Claim 1 Intersection of an infinite family of basic closed subsets of a Cartesian power of $\tilde{\mathrm{V}}$ is equal to the intersection of its finite subfamily.

Indeed, since for a given set of variables there are finite number of ways to partition (enumerate) the variables as $\{v_{ij} : i \leq s,\ j \leq r_i\}$, we may assume that all core formulas defining the members of the family have the same partition of variables. Now by Lemma 2 and Lemma 3(i) of 3.5 the intersection of sets defined by $\exists e\ R_\alpha$, $\alpha \in I$, reduces to the intersection of Zariski closed sets defined by R_α, $\alpha \in I$, which obviously stabilises.

Using Koenig's Lemma we get

Claim 2 The A-topology is Noetherian.

Since for $s+u=0$ a general core formula $\exists e\, R$ takes the form $R(x,y)$ the following is obvious.

Claim 3 The restriction of the A-topology to the sort F is the classical Zariski topology.

Claim 4 Any definable subset of a Cartesian power of $\tilde{\mathrm{V}}$ is equal to the Boolean combination of closed subsets, that is, is **constructible**.

Indeed, by the Corollary in 3.3 it is sufficient to prove the statement for subsets defined by general core formulas. The Corollary in 3.5 together with Lemmas 3(ii) provide the rest.

We will also need a more detailed presentation of sets obtained by projecting closed sets onto coordinate subspaces, as well as fibers of these projections.

Lemma 1 *Let $\exists e\, R$ be the general core formula in the notation of 3.3 and $a \in \{1, \ldots, s+u\}$, $b \in \{1 \ldots, r_a\}$ some indices. Then the formula $\exists v_{ab} \exists e\, R$ is equivalent to a general core formula $\exists e'\, R'$ with the kernel R' equivalent to one of the following*

$$\begin{array}{ll} (i) & \exists y_{a-s,b1} \ldots y_{a-s,bN} R, \\ (ii) & \exists z_{ab1} \ldots z_{abN} R \quad \textit{or} \\ (iii) & \exists m_a \exists \{t_{akjl} : k \leq d,\ j,l \leq N\} \exists z_{ab1} \ldots z_{abN} R. \end{array}$$

Proof (i) Suppose $s < a \leq s+u$. Since v_{ab} does not occur in $A(e,m,t,z,v)$

and $R(m,t,x,y,z)$, the formula $\exists v_{ab} \exists e\, R$ is equivalent to

$$\exists\ e_1, \dots e_s \exists\ \dots$$

$$A(e,m,t,z,v) \&\ (\exists v_{ab} B(\hat{e},y,v))\ \&\ R(m,t,x,y,z),$$

with the quantifier prefix the same as in $\exists e\, R$. Looking at the form of $B(\hat{e},y,v)$ one sees that $(\exists v_{ab} B(\hat{e},y,v))$ is equivalent to some $B(\hat{e},y',v')$ with $y' = y \setminus \{y_{a-s,b1} \dots y_{a-s,bN}\}$ and $v' = v \setminus \{v_{ab}\}$. Now we can equivalently rewrite the formula as

$$\exists\ e_1, \dots e_s \exists\ \dots$$

$$A(e,m,t,z,v) \&\ B(\hat{e},y',v') \&\ \exists y_{a-s,b1} \dots y_{a-s,bN} R(m,t,x,y,z),$$

where $\exists y_{a-s,b1} \dots y_{a-s,bN}$ moved from the quantifier prefix to the end of the formula. Of course, by quantifier elimination in algebraically closed fields, $\exists y_{a-s,b1} \dots y_{a-s,bN} R$ is a constructible predicate.

(ii) and (iii). Suppose $a \le s$. Then the formula $\exists v_{ab} \exists e\, R$ is equivalent to

$$\exists\ e_1, \dots e_s \exists\ \dots$$

$$(\exists v_{ab} A(e,m,t,z,v))\ \&\ B(\hat{e},y,v) \&\ R(m,t,x,y,z).$$

One can obviously eliminate the quantifier from $\exists v_{ab} A(e,m,t,z,v)$ by substituting v_{ab} everywhere in $A(e,m,t,z,v)$ by the term $\sum_{\ell \le N} z_{ab\ell} e_a(\ell)$. This makes the conjunct $v_{ab} = \sum_{\ell \le N} z_{ab\ell} e_a(\ell)$ in $A(e,m,t,z,v)$ a tautology and after removing it we get a formula without v_{ab} which in the case $r_a \neq 1$ is again of the form $A(e,m,t,z',v')$, where $v' = v \setminus \{v_{ab}\}$, $z' = z \setminus \{z_{ab1} \dots z_{abN}\}$. This gives us a general core formula of the form (ii) for $\exists v_{ab} \exists e\, R$.

In case $r_a = 1$ the conjunct $v_{ab} = \sum_{\ell \le N} z_{ab\ell} e_a(\ell)$ is the only one that uses the variables e_a. By eliminating this conjunct we made other subformulas containing e_a redundant. So we eliminate

$$\bigwedge_{j \le N} E(e_a, m_a)\ \&\ P^A(\{t_{akn\ell}\}_{k \le d,\ \ell,n \le N}; m_a) = 0\ \&\ \bigwedge_{k \le d, j \le N} \mathbf{U}_k e_a(j) = \sum_{\ell \le N} t_{akj\ell} e_a(\ell)$$

from $A(e,m,t,z,v)$ as well (notice that by our assumptions $P^A(\{t_{akn\ell}\}_{k \le d,\ \ell,n \le N}; m_a) = 0$ is also copied in R). The resulting formula is again of the form $A(e',m',t',z',v')$, with $v' = v \setminus \{v_{ab}\}$, $m' = m \setminus \{m_a\}$, $z' = z \setminus \{z_{ab1} \dots z_{abN}\}$ and $t' = t \setminus \{t_{akjl} : k \le d,\ j,\ell \le N\}$. Now we may push the quantifiers $\exists m_a$, $\exists \{z_{ab1} \dots z_{abN}\}$ and $\exists \{t_{akj\ell} : k \le d,\ j,\ell \le N\}$ to the end of the formula and get the general core formula of the form (iii).□

Lemma 2 *Suppose* $\exists e\, R$ *does not contain* x*, free variables of the sort* F*. Let* $\exists e' R'$ *be the general core formula equivalent to* $\exists v_{ab} \exists e\, R$ *as in the above Lemma. More precisely* $R' = \exists u R(u, w)$*, some* u *depending on the case. Let* $v' = v \setminus \{v_a b\}$ *and* $\hat{v}'$ *is a tuple in* $\tilde{\mathrm{V}}$ *satisfying* $\exists e' R'$*. Then* $(\exists e R)^{v' := \hat{v}'}$ *has kernel of the form* $R(u, \hat{w})$*, for some* $\hat{w}$*.*

Proof Follow the analysis in the proof of Proposition in 3.3. In case (i) the substituition $v' := \hat{v}'$ fixes the whole of e, m, t, z and $y \setminus \{y_{a-s,b,1} \dots, y_{a-s,b,N}\}$, so the kernel is $R^{m=\hat{m}, t=\hat{t}, z=\hat{z}, y'=\hat{y}'}$. In other words, in this case we satisfied the requirement of the Lemma with $u = (y_{a-s,b,1} \dots, y_{a-s,b,N})$ and $w = (m, t, z, y')$.

In case (ii) again $\hat{v}'$ fixes the whole of e, m, t, y and $z \setminus \{z_{ab1} \dots z_{abN}\}$. In case (iii) $\hat{v}'$ fixes $e \setminus e_a$ $m \setminus m_a$, $t \setminus t_a$, y and $z \setminus \{z_{ab1} \dots z_{abN}\}$.□

4.2 For further purposes we need a more detailed understanding of intersections of closed sets.

Let $\{v_1, \dots, v_n\}$ be a linear reenumeration of variables $\{v_{ij} : i \le s, j \le r_i\}$, $n = r_1 + \dots + r_s$, of sort $\tilde{\mathrm{V}}$ in a general core formula $\exists e R$ (the variables $\{v_{ij} : s < i \le s + u, j \le r_i\}$ remain unchanged). We write $k \sim_R k'$ for $k, k' \in \{1, \dots, n\}$ if k and k' correspond to some (i, j) and (i, j') in the old enumeration. This is an equivalence relation. We denote I_R the subset $\{1, \dots, n\}$ corresponding to $\{(i, 1) : i = 1, \dots, s\}$ in the partitioning enumeration, the set of representatives of $\sim_R$-classes.

We use the abbreviation e_i for $\{e_i(1), \dots, e_i(N)\}$, t_i for $\{t_{ikj\ell} : k \le d;\ j, l \le N\}$ and z_i for $\{z_{ij\ell} : j \le r_i;\ \ell \le N\}$, $i = 1, \dots, n$, along with other obvious abbreviations. In particular, $\mathbf{U} e_i = t_i e_i$ stands for

$$\bigwedge_{k \le d, j \le N, i \le s} \mathbf{U}_k e_i(j) = \sum_{\ell \le N} t_{ikj\ell} e_i(\ell).$$

We rewrite equivalently the general core formula $\exists e\, R$ of 3.3 as $\tilde{R}(v, x)$:

$$\exists\ e_1, \dots e_n\ \exists\ t_1, \dots, t_n,\ \exists\ z_1, \dots, z_n\ y\ \exists\ m_1, \dots m_n$$

$$\bigwedge_{i \le n} E(e_i, m_i)\ \&\ \mathbf{U} e_i = t_i e_i\ \&\ v_i = z_i e_i\ \&\ \bigwedge_{i \sim_R j} e_i = e_j\ \&\ B(\hat{e}, y, v)$$

$$\& \ R(m,t,z,x,y) \ \& \ \bigwedge_{i \sim_R j} m_i = m_j \ \& \ t_i = t_j.$$

This is not a core formula because of the component $\bigwedge_{i \sim_R j} e_i = e_j$.

Remark 1 In R only m_i, t_i with $i \in I_R$ as well as z_i, $i \leq n$, y and x occur explicitly. Let $d_R = \dim R$, the dimension of the variety defined by R in the space given by these variables. Obviously we may assume that R depends on all variables m_i, t_i, z_i, $i \leq n$, y and x. Then in the bigger ambient space we still have

$$d_R = \dim[R \& \bigwedge_{i \sim j} m_i = m_j \ \& \ t_i = t_j].$$

Let $\exists e\, S$ be another general core formula of the same variables with possibly different partitioning enumeration $\{v_{ij} : i \leq s' + u, j \leq r'_i\}$. We assume that variables $\{v_{ij} : s' < i \leq s' + u, j \leq r'_i\}$ and parameters $\hat{e}_{s'+1}, \ldots, \hat{e}_{s'+u}$ are the same in both $\exists e\, R$ and $\exists e\, S$.

We re-enumerate the variables $\{v_{ij} : i \leq s', j \leq r'_i\}$ linearly as $v_1, \ldots, v_n$. We have the corresponding equivalence relation $\sim_S$ on $\{1, \ldots, n\}$ and a set of its representatives I_S. As above $\exists e S$ can be equivalently rewritten as the formula $\tilde{S}(v,x)$:

$$\exists \ e'_1, \ldots e'_n \ \exists \ t'_1, \ldots, t'_n, \ \exists \ z'_1, \ldots, z'_n \ y \ \exists \ m_1, \ldots m_n$$

$$\bigwedge_{i \leq n} E(e'_i, m_i) \ \& \ \mathbf{U}e'_i = t'_i e'_i \ \& \ v_i = z'_i e'_i \ \& \ \bigwedge_{i \sim_S j} e'_i = e'_j \ \& \ B(\hat{e}, y, v)$$

$$S(m', t', x, y, z') \ \& \ \bigwedge_{i \sim_S j} m_i = m_j \ \& \ t'_i = t'_j.$$

Lemma The formula $\tilde{R}(v,x) \ \& \ \tilde{S}(v,x)$ is equivalent to the formula $\tilde{T}(v,x)$:

$$\exists \ e_1, \ldots e_n \ \exists \ t_1, \ldots, t_n, \ \exists \ z_1, \ldots, z_n \ y \ \exists \ m_1, \ldots m_n$$

$$\bigwedge_{i \leq n} E(e_i, m_i) \ \& \ \mathbf{U}e_i = t_i e_i \ \& \ v_i = z_i e_i \ \& \ \bigwedge_{i \sim_{RS} j} e_i = e_j \ \& \ B(\hat{e}, y, v)$$

$$R(m,t,x,y,z) \& S(m,t,x,y,z) \ \& \ \bigwedge_{i \sim_{RS} j} m_i = m_j \ \& \ t_i = t_j,$$

where $\sim_{RS}$ is the transitive closure of the composition of the two equivalence relations $\sim_R$ and $\sim_S$.

Proof The implication $\tilde{T}(v,x) \to \tilde{R}(v,x)$ & $\tilde{S}(v,x)$ is obvious.

For converse suppose $\tilde{R}(v,x)$ & $\tilde{S}(v,x)$ holds. This implies the existence of $e_i, e'_i, t_i, t'_i, z_i, z'_i, m_i$ $(i = 1, \ldots, n)$ and y which satisfy

$$\bigwedge_{i \le n} E(e_i, m_i) \ \& \ \mathbf{U}e_i = t_i e_i \ \& \ v_i = z_i e_i \ \& \ \bigwedge_{i \sim_R j} e_i = e_j \ \& \ B(\hat{e}, y, v)$$

$$R(m, t, x, y, z) \ \& \ \bigwedge_{i \sim_R j} m_i = m_j \ \& \ t_i = t_j$$

and

$$\bigwedge_{i \le n} E(e'_i, m_i) \ \& \ \mathbf{U}e'_i = t'_i e'_i \ \& \ v_i = z'_i e'_i \ \& \ \bigwedge_{i \sim_S j} e'_i = e'_j \ \& \ B(\hat{e}, y, v)$$

$$S(m, t', x, y, z') \ \& \ \bigwedge_{i \sim_S j} m_i = m_j \ \& \ t'_i = t'_j.$$

m_i must be the same in both formulas since $m_i = \pi(v_i)$. It follows from the assumption 2.1.4 that for some $\gamma_i \in \Gamma_m$, $e_i = \gamma_i e'_i$. Since R is Γ-invariant we can exchange, for $i \in I_R$, e_i by $\gamma_i e_i$, t_i by $t_i^{\gamma_i}$ and z_i by $z_i \gamma_i$ without changing the validity of R and so may assume that $\gamma_i = 1$ and $e_i = e'_i$ for $i \in I_R$.

By symmetry we can reduce to the situation that also $e_j = e'_j$ for $j \in I_S$.

Claim We can choose $e_i = e'_i$ for all $i \le n$.

Proof. By induction on n. We have already $e_i = e'_i$ for all $i \in I_R \cup I_S$, so we assume that $I_R \cup I_S \subseteq \{1, \ldots, n-1\}$ and we can choose $e_i = e'_i$ for all $i \le n-1$. We have $e_n = e_\ell = e'_\ell$ for some $\ell \in I_R$, $\ell \sim_R n$, and $e'_n = e'_k = e_k$, for some $k \in I_S$, $k \sim_S n$. From the equivalences it follows that $m_i = m_n = m_k$, i.e. the modules coincide. So, $e_n = \gamma e'_n$ for some $\gamma \in \Gamma_m$.

Let $J_k = \{i \le n : e_i = e_k\}$, $J'_k = \{i \le n : e'_i = e_k\}$. Note that $n \notin J_k$. Apply the substitution $e_i \mapsto \gamma e_i$ and $e'_j \mapsto \gamma e'_j$ for all $i \in J_k$, $j \in J'_k$, leaving e_i and e'_j for $i \notin J_k$, $j \notin J'_k$ unchanged in

$$\bigwedge_{i \le n} E(e_i, m_i) \ \& \ \mathbf{U}e_i = t_i e_i \ \& \ v_i = z_i e_i \ \& \ \bigwedge_{i \sim_R j} e_i = e_j$$

and

$$\bigwedge_{i \le n} E(e'_i, m_i) \ \& \ \mathbf{U}e'_i = t'_i e'_i \ \& \ v_i = z'_i e'_i \ \& \ \bigwedge_{i \sim_S j} e'_i = e'_j.$$

This induces the correspondent transformation of t, t', z, z' which by Γ-invariance does not change the validity of $R(m, t, x, y, z)$ and $S(m, t', x, y, z')$.

This preserves all the existing equalities and gives $e'_n = e_n$ for the new value of e'_n. Claim proved.

This brings us to the situation with $e_i = e'_i$, $t_i = t'_i$ and $z_i = z'_i$ in the formulas above. Thus

$$\bigwedge_{i\leq n} E(e_i, m_i) \ \&\ \mathbf{U}e_i = t_i e_i \ \&\ v_i = z_i e_i \ \&\ \bigwedge_{i\sim_{RS} j} e_i = e_j$$

$$R(m, t, z, x)\&S(m, t, z, x) \ \&\ \bigwedge_{i\sim_{RS} j} m_i = m_j \ \&\ t_i = t_j,$$

hold. This proves the converse implication.□

Corollary 1 *The intersection of two basic closed sets given by general core formulas $\exists e\, R$ and $\exists e\, S$ with arbitrary partitioning enumerations and the same parameters $\hat{e}_{s+1}, \ldots, \hat{e}_{s+u}$ is a basic closed set given by a core formula $\exists e\, T$, with T equivalent to*

$$R(m, t, x, y, z)\&S(m, t, x, y, z) \ \&\ \bigwedge_{i\sim_{RS} j} m_i = m_j \ \&\ t_i = t_j.$$

Indeed, $\tilde{T}(v, x, y)$ can be transformed into a core formula by the following process.

Let $I_{RS} \subseteq \{1, \ldots, n\}$ be a set of representatives of $\sim_{RS}$-classes. Assuming $I_{RS} \subseteq \{1, \ldots, u\}$ re-enumerate $v_1, \ldots, v_n$ as $v_{ij} : i \leq u, j \leq r_i\}$, v_{i1} is the v_i in the linear enumeration and indices (ij) correspond to indices equivalent to i by $\sim_{RS}$.

Using the equalities $e_i = e_j$, $m_i = m_j$ and $t_i = t_j$, for $i \sim_{RS} j$, we delete e_j, m_j and t_j, with $j > u$, everywhere from $\tilde{T}$ along with the subformulas stating the equalities.

We re-enumerate z-variables in accordance with enumeration v_{ij}, so that now the formula $\tilde{T}$ says now that $v_{ij} = z_{ij} e_i$ for every $i \leq u$ and $j \leq r_i$.

After that $\tilde{T}$ transforms to

$$\exists e_1, \ldots, e_u \ \exists t_1, \ldots, t_u \ \exists m_1, \ldots, m_u \ \exists z_{11}, \ldots, z_{ur_u}$$

$$\bigwedge_{i\le u} E(e_i,m_i) \ \&\ \mathbf{U}e_i = t_ie_i \ \&\ \bigwedge_{j\le r_u} v_{ij} = z_{ij}e_i \ \&\ B(\hat{e},y,v)$$

$$R'(m,t,x,y,z)\&S'(m,t,x,y,z),$$

where R' and S' obtained by substituting t_i and m_i instead of t_j and m_j, for $j \sim_{RS} i$, $j > u$. This is a general core formula.

Remark 2 By the Remark in 3.3 we can always assume that parameters $\hat{e}$ in both formulas are the same.

Combining Corollary 1 with (i) and (ii) of the definition 4.1 we get.

Corollary 2 *Every closed set in the* A*-topology is equal to the union of a finite family of closed sets each of the form* $\hat{P}$*, for* P *a Zariski closed predicate.*

Corollary 3 *Given a basic closed set* $\mathbf{P} \subseteq \tilde{\mathrm{V}}^n \times \mathrm{F}^p$*, there is a core formula* $\exists e\, P$ *defining* $\mathbf{P}$ *with the finest partition of the* v*-variables. That is, for every* $\exists e\, R$ *defining the same set the partition* $\sim_P$ *is refining* $\sim_R$ *.*

Fixing a choice of parameters $\hat{e}$ *(one of the finitely many), the Zariski closed relation* P *above is determined uniquely by the set* $\mathbf{P}$*. Any other choice* $\hat{e}'$ *of parameters for* $\mathbf{P}$ *determines a Zariski closed relation* P' *obtained from* P *by a linear transformation in variables* y*.*

Indeed, take for $\exists e\, P$ the formula obtained by taking the conjunction of all possible representations $\exists e\, R$ of $\mathbf{P}$, R Zariski closed, using Corollary 1.

Lemma 4 in 3.5 implies the uniqueness of P.$\square$

We will say that the algebraic constructible set $P(\mathrm{F})$ for P and $\mathbf{P}$ as above is **associated** with $\mathbf{P}$. If $P(\mathrm{F})$ is Zariski closed we call $P(\mathrm{F})$ **the variety associated** with the closed set $\mathbf{P}$.

4.3 From now on when we write a basic closed set in the form $\hat{P}$ (equivalently, use the core formula $\exists e\, P$) the kernel P is canonical, that is is uniquely determined by the set $\hat{P}$.

We define

$$\dim \hat{P} := \dim P(\mathrm{F}),$$

where dim on the right is the dimension of the algebraic variety.

For a constructible set S we define

$$\dim S := \dim \bar{S}, \text{ where } \bar{S} \text{ is the closure of } S.$$

Suppose $v = v_1^\frown v_2$, $|v_1| = n_1$, $|v_2| = n_2$, $x = x_1^\frown x_2$, $|x_1| = k_1$, $|x_2| = k_2$ and let

$$\mathrm{pr} : \tilde{\mathrm{V}}^{n_1+n_2} \times \mathrm{F}^{k_1+k_2} \to \tilde{\mathrm{V}}^{n_1} \times \mathrm{F}^{k_1}$$

be the projection $\mathrm{pr} : v_1^\frown v_2^\frown x_1^\frown x_2 \mapsto v_1^\frown x_1$.

Proposition *Let* $S \subseteq \tilde{\mathrm{V}}^{n_1+n_2} \times \mathrm{F}^{p_1+p_2}$ *be a closed set. Then*
(i) $\mathrm{pr}(S)$ *is a constructible set;*
(ii) for each $a \in \mathrm{pr}(S)$, *the set* $S \cap \mathrm{pr}^{-1}(a)$ *is closed;*
(iii) for each nonnegative integer ℓ *the set*

$$\{a \in \mathrm{pr}(S) : \ \dim S \cap \mathrm{pr}^{-1}(a) \geq \ell\}$$

is constructible. If $\ell > \min_{a \in \mathrm{pr}(S)} \dim S \cap \mathrm{pr}^{-1}(a)$ *then the set is contained in a proper subset closed in* $\mathrm{pr}(S)$.
(iv) assuming S *is irreducible, we have*

$$\dim S = \dim \mathrm{pr}(S) + \min_{a \in \mathrm{pr}\, S} \dim S \cap \mathrm{pr}^{-1}(a).$$

(v) for any two irreducible $S_1, S_2 \subseteq \tilde{V}^n \times \mathrm{F}^p$, *for every irreducible component* S_0 *of* $S_1 \cap S_2$,

$$\dim S_0 \geq \dim S_1 + \dim S_2 - \dim \tilde{V}^n \times \mathrm{F}^p.$$

Proof (i) Follows from Claim 4 in 4.1.
(ii) Just notice that

$$S \cap \mathrm{pr}^{-1}(a) = S \cap \{v_1^\frown x_1 = a\}$$

and notice that $\{v_1^\frown x_1 = a\}$ is a basic closed set.
(iii) By our definition of dimension and the two Lemmas in 4.1 this is equivalent to the same statement for S an affine algebraic variety. This is a well-known theorem for algebraic varieties used as an axiom (FC) for Zariski geometries in [Z1].
(iv) First observe

Claim If S is irreducible then $S = \hat{P}$ with the associated variety P of the form

$$\bigcup_{\gamma \in \Gamma^s} R^\gamma, \quad R \text{ irreducible.}$$

Indeed, by definition S is a union of sets of the form $\hat{P}$. Since it is irreducible there is just one such in the union. Let $R(\mathrm{F})$ be an irreducible component of the variety $P(\mathrm{F})$. By Γ-invariance $R^\gamma(\mathrm{F})$ is also a component of P, for all $\gamma \in \Gamma_P$. By irreduciblity of S the union of all R^γ is equal to P. Claim proved.

Again as in (iii), by 4.1, (iv) is equivalent to

$$\dim P = \dim \mathrm{pr}\, P + \min_{b \in \mathrm{pr}\, P} \dim P \cap \mathrm{pr}^{-1}(b)$$

for an appropriate projection pr. But this is the known *addition formula* for algebraic varieties and more generally Zariski structures, see [Z1].

(v) First observe that by the Claim above irreducible S_i $(i = 0, 1, 2)$ have to be of the form $\hat{P}_i$, for P_i of the form $\bigvee_{\gamma \in \Gamma^s} R_i^\gamma$, R_i irreducible.

The rest follows from Corollary 1 of 4.2 (see also Remark 1 in the same section). In the present notation we get by the Corollary that the kernel in $\hat{P}_1 \cap \hat{P}_2$ corresponds to the intersection of two algebraic subvarieties of dimensions $\dim P_1$ and $\dim P_2$ in the ambient affine space of dimension $\dim \tilde{\mathrm{V}}^n + \dim \mathrm{F}^p$. By the Dimension Theorem for affine spaces we get the required inequality. $\square$

Theorem *For any algebra* A *satisfying the assumptions 2.1(1-4) the structure* $\tilde{\mathrm{V}}$ *is a Zariski geometry, satisfying the presmoothness condition provided the affine algebraic variety* V *is smooth.*

Proof The Proposition above together with the topological subsection 4.1 proves all the assumptions defining Zariski geometries, see [Z1].$\square$

References

[BG] K.Brown and K.Goodearl, **Lectures on Algebraic Quantum Groups**, Birkhauser, 2002

[C] A.Connes, **Noncommutative geometry**, Academic Press 1994

[H] E.Hrushovski, *Groupoids, imaginaries and internal covers,* arXiv:math.LO/0603413

[HZ] E.Hrushovski and B.Zilber, *Zariski Geometries,* Journal of AMS, 9(1996), 1-56

[Man] Yu.Manin, **Topics in Noncommutative geometry,** Princeton Univ. Press, Princeton, 1991

[Maj] Sh.Majid, **A quantum groups primer**, LMS Lecture Notes Ser. 292, Cambridge University Press, 2002

[P] B.Poizat, *L'egalite au cube,* JSL 66,(2001), 1647-1676

[Sk] Z.Skoda, *Noncommutative localization in noncommutative geometry* arXiv:math.QA/0403276

[RVW] N.Reshetikhin, A.Voronov, A.Weinstein, *Semiquantum geometry,* arXiv:q-alg/9606007

[Z1] B.Zilber, *Lecture notes on Zariski structures*, 1994-2004, web-page

[Z2] B.Zilber, *Non-commutative geometry and new stable structures*, Newton Institute preprints series NI05048-MAA, 2005